Exceptional field theory and half-maximal SUSY

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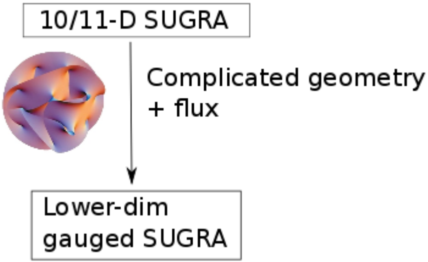
From lower dimensions to 10/11-D



- I have some nice lower-dim SUGRA with e.g. Lifshitz background, black holes, AdS vacuum, ...
- Does it come from 10/11-D SUGRA (string theory)?
- SUGRA specified by no. of SUSY, multiplets + "embedding tensor": $\theta_{\bar{M}\bar{N}}{}^{\bar{P}}$.
  [de Wit, Nicolai, Samtleben, . . .]
- Can we "geometrise" $\theta_{\bar{M}\bar{N}}{}^{\bar{P}}$?

Consistent truncation?

Consistent truncation

A *consistent* truncation of 10/11-d SUGRA is an Ansatz that yields a lower-dimensional (gauged) SUGRA, all of whose solutions are also solutions of the initial SUGRA.

- E.g. GR on S^1 has a consistent truncation to Einstein-Maxwell-Dilaton.
- GR on S^1 with $\phi = 0$ is *not* consistent!

$$\Box\phi \sim F^2 \,.$$

Consistent truncations are hard

- Non-linearity of EoM's $\Rightarrow$ consistent truncations are in general hard to find!
- Useful tools: singlets under symmetry group, Scherk-Schwarz reductions on group manifolds.
- Many interesting consistent truncations do not fall into these classes:
  - IIB on S^5,
  - 11-D on S^4, S^7.
  - $\mathcal{N} = 1$ on $G \longrightarrow$ gauge group $G \times G$.

Maximally SUSY truncations

- Consider maximally SUSY truncations of $(D+n)$-dimensional SUGRA (no fluxes), $X^{\hat{\mu}} = (x^\mu, y^m)$
- $(D+n)$-dimensional KK Ansatz:

$$\mathcal{E}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} \phi^{-\gamma} e_\mu{}^a & A_\mu{}^m \Phi_m{}^i \\ 0 & \Phi_m{}^i \end{pmatrix}$$

- Maximal SUSY $\Rightarrow$ maximal set of well-defined nowhere-vanishing spinors
- $\Rightarrow$ "parallelisable internal manifold" ("Id structure")
- $\Rightarrow$ n well-defined nowhere-vanishing vector fields $\Phi_i{}^m \in \mathrm{GL}(n)$
- Internal vielbein: $g_{mn} = \Phi_m{}^i \Phi_{n\,i}$.

Scherk-Schwarz truncations

- $D + n$-dimensional GR:

$$\mathcal{E}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} \phi^{-\gamma} e_\mu{}^a & A_\mu{}^m \Phi_m{}^i \\ 0 & \Phi_m{}^i \end{pmatrix}$$

- Simplest example of consistent truncation: T^n

$$\begin{aligned} e_\mu{}^a(x, y) &= e_\mu{}^a(x)\,, \\ A_\mu{}^m(x, y) &= A_\mu{}^m(x)\,, \\ \Phi_m{}^i(x, y) &= \Phi_m{}^i(x)\,. \end{aligned}$$

- Obtain ungauged D-dimensional Einstein-Maxwell-Scalar
- $U(1)^n$ gauge group.

Scherk-Schwarz truncations

- $D+n$-dimensional GR:

$$\mathcal{E}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} \phi^{-\gamma} e_\mu{}^a & A_\mu{}^m \Phi_m{}^i \\ 0 & \Phi_m{}^i \end{pmatrix}$$

- More interesting: Scherk-Schwarz Ansatz on group manifold G

$$\begin{aligned} e_\mu{}^a(x,y) &= e_\mu{}^a(x)\,, \\ A_\mu{}^m(x,y) &= A_\mu{}^{\bar{m}}(x)\, U_{\bar{m}}{}^m(y)\,, \\ \Phi_m{}^i(x,y) &= \Phi_{\bar{m}}{}^i(x)\, (U^{-1})_m{}^{\bar{m}}(y)\,, \end{aligned}$$

where $U \in G \subset \mathrm{GL}(n)$.

- Obtain gauged D-dimensional Einstein-Maxwell-Scalar
- Gauge group $G \subset \mathrm{GL}(n)$ ("Intrinsic torsion"):

$$L_{U_{\bar{m}}} U_{\bar{n}}{}^m = f_{\bar{m}\bar{n}}{}^{\bar{p}} U_{\bar{p}}{}^m\,.$$

Beyond Scherk-Schwarz

- Many important truncations are not on group manifolds!
- Cannot be explained by simple group theory arguments
  - 11-d on S^4, $S^7 \longrightarrow \mathrm{SO}(5)$ / $\mathrm{SO}(8)$ gSUGRA?
  - IIB on $S^5 \longrightarrow \mathrm{SO}(6)$ gSUGRA?
- $\mathcal{N} = 1$ on $G \longrightarrow G \times G$ gauge group!
- Important for AdS/CFT, etc.
- *p*-form gauge fields are *crucial*.

Generalising Scherk-Schwarz via Kaluza-Klein

- Consider 1-form.
- Einstein-Maxwell-Dilaton: $S = \int d^{D+2}x\sqrt{|g|}\left(R - (\nabla\phi)^2 - e^{\alpha\phi}F^2\right)$
- For specific α has consistent S^2 reduction [Cvetic, Lü, Pope].

$$ds^2_{D+2} = Y^{\frac{1}{D}}\left(\Delta^{\frac{1}{D}}ds^2_D + g^{-2}\Delta^{-\frac{D-1}{D}}T^{-1}_{ij}\mathfrak{D}\mu^i\mathfrak{D}\mu^j\right),$$

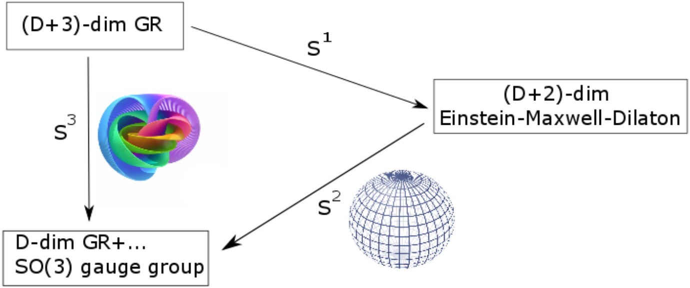
$$e^{\sqrt{\frac{2(D)}{D+1}}\hat{\phi}} = \Delta^{-1}Y^{\frac{D-1}{D+1}},$$

$$\hat{F}_2 = \frac{1}{2}\epsilon_{ijk}\left(g^{-1}\Delta^{-2}\mu^i\mathfrak{D}\mu^j\wedge\mathfrak{D}\mu^k - 2g^{-1}\Delta^{-2}\mathfrak{D}\mu^i\wedge\mathfrak{D}T_{jl}T_{km}\mu^l\mu^m - \Delta^{-1}F^{ij}_{(2)}T_{kl}\mu^l\right).$$

- D-dimensional GR-YM-Scalars with $\mathrm{SU}(2)$ gauge group!
- Better way to see this?

Consistent S^2 reductions

- $(D+3)$-dim GR: $S = \int d^{d+3}\sqrt{|G|}R(G)$

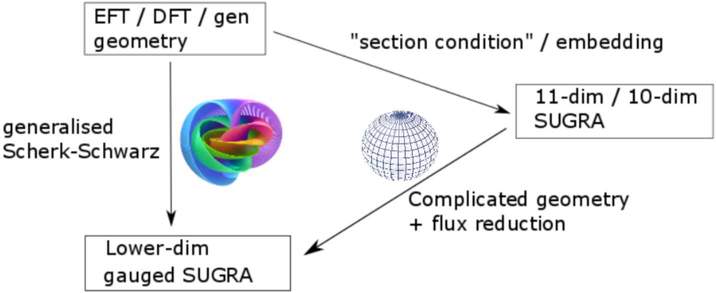


- Scherk-Schwarz on $\mathrm{SU}(2)$ [Cvetic, Lü, Pope, Gibbons].
- Explains existence of consistent truncation, value of α and truncation Ansatz:

$$
\begin{aligned}
e_\mu{}^a(x,y) &= e_\mu{}^a(x)\,, \\
A_\mu{}^m(x,y) &= A_\mu{}^{\bar{m}}(x)\, U_{\bar{m}}{}^m(y)\,, \\
\Phi_m{}^i(x,y) &= \Phi_{\bar{m}}{}^i(x)\, (U^{-1})_m{}^{\bar{m}}(y)\,.
\end{aligned}
$$

Exceptional field theory

- p-form fluxes are crucial in S^4, S^5, S^7 compactifications of 10/11-d SUGRA.
- "Geometrise" these via *exceptional field theory / generalised geometry*.



Exceptional field theory

[Siegel]; [Duff]; [Hull, Zwiebach]; [Hohm, Hull, Zwiebach]; [Hull]; [Berman, Perry], [Hohm, Samtleben]; [Hitchin]; [Gualtieri]; [Coimbra, Strickland-Constable, Waldram]

- KK split of 11-d SUGRA: $M_{11} = M_D \times M$
- Unify symmetries of fields in M:

$$\delta g = L_v g \,, \qquad \delta C_{(3)} = L_v C_{(3)} + d\chi_{(2)} \,,$$
$$\delta C_{(6)} = L_v C_{(6)} + d\chi_{(5)} - \frac{1}{2} d\chi_{(2)} \wedge C_{(3)} \,, \ldots$$

- Generalised vector field: $V = v + \chi_{(2)} + \chi_{(5)} + \ldots$
- Section of $\mathcal{R}_1 \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus \ldots$
- $\mathcal{R}_1$ admits action of $E_{d(d)}$.

Examples

D	$E_{d(d)}$	H_d
9	$\mathrm{SL}(2) \times \mathbb{R}^+$	$\mathrm{U}(1)$
8	$\mathrm{SL}(2) \times \mathrm{SL}(3)$	$\mathrm{U}(1) \times \mathrm{SU}(2)$
7	$\mathrm{SL}(5)$	$\mathrm{USp}(4)$
6	$\mathrm{Spin}(5,5)$	$\mathrm{USp}(4) \times \mathrm{USp}(4)$
5	$E_{6(6)}$	$\mathrm{USp}(8)$
4	$E_{7(7)}$	$\mathrm{SU}(8)$
3	$E_{8(8)}$	$\mathrm{SO}(16)$

- $\mathcal{R}_1 \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus \ldots$
- $D = 7$, $\mathrm{SL}(5)$, $R_1 = \mathbf{10} = 4 + 6$.
- $D = 5$, $E_{6(6)}$, $R_1 = \mathbf{27} = 6 + 15 + 6$.
- Patching with $\mathrm{GL}(d) \subset E_{d(d)}$.

Degrees of freedom

- Recall $M_{11} = M_D \times M_d$.
- Internal bosonic degrees of freedom (scalars on M_D) $\longrightarrow$ generalised metric $\mathcal{M}_{MN}$.
- $\mathcal{M}_{MN}$ parameterises coset $E_{d(d)}/H_d$:

$$\{g\,,\ C_{(3)}\,,\ C_{(6)}\,,\ \ldots\} = \mathcal{M}_{MN} \in \frac{E_{d(d)}}{H_d}\,.$$

- Covectors on M_D, ... also form generalised tensors

$$\{g_{\mu\, i}\,,\ C_{\mu\, ij}\,,\ \ldots\} = \mathcal{A}_\mu{}^M \in \tilde{\Omega}_1\left(M_D\right) \times \Gamma\left(\mathcal{R}_1\right)\,,$$
$$\{C_{\mu\nu\, i}\,,\ \ldots\} = \mathcal{B}_{\mu\nu\, I} \in \tilde{\Omega}_2\left(M_D\right) \times \Gamma\left(\mathcal{R}_2\right)\,.$$

- Obtain other generalised vector bundles: $\mathcal{R}_2$, $\mathcal{R}_3$, ... this way.
- Fermions organise according to H_d.

Generalised Lie derivative and action

- Generalised Lie derivative

$$\mathcal{L}_V = V^M \partial_M + (\partial \times_{adj} V) = \text{diffeo} + \text{gauge transf}\,,$$

with $\partial_M = \left(\partial_i\,,\ \partial^{ij}\,,\ \partial^{i_1\ldots i_5}\,,\ldots\right) = (\partial_i\,,\ 0\,,\ \ldots\,,\ 0)$.
- E.g.

$$\mathcal{L}_V \mathcal{M}_{MN} \longrightarrow \left\{L_v g\,,\ L_v C_{(3)} + d\chi_{(2)}\,,\ \ldots\right\}\,.$$

- Can construct generalised connections, curvature, . . . , and action!

$$L = \sqrt{|g|}\left(R_g + g^{\mu\nu}\partial_\mu \mathcal{M}^{MN}\partial_\nu \mathcal{M}_{MN} + F_{\mu\nu}{}^M F^{\mu\nu\,N}\mathcal{M}_{MN} - \mathcal{R} + \ldots\right)\,,$$

with

$$\mathcal{R} = \mathcal{M}^{MN}\mathcal{R}_{MN} = \mathcal{M}^{MN}\partial_M \mathcal{M}^{PQ}\partial_N \mathcal{M}_{PQ} + \ldots$$

- General ∂_M?
  Closure: $[\mathcal{L}_V, \mathcal{L}_W] = \mathcal{L}_{[V,W]} \Rightarrow$ "section condition"

$$Y^{MN}_{PQ} \partial_M f \partial_N g = Y^{MN}_{PQ} \partial_M \partial_N f = 0 \,. \qquad (1)$$

- Different solutions of (1) give 11-d/IIA/IIB generalised geometries.
- $\mathrm{GL}(d) \subset E_{d(d)}$:

$$\partial_M = \left(\partial_i \,,\; \partial^{ij} \,,\; \partial^{i_1 \ldots i_5} \,, \ldots\right) = \left(\partial_i \,,\; 0 \,,\; \ldots \,,\; 0\right) \,.$$

- 11-d gen geometry: $\mathcal{R}_1 \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus \ldots$
- $\mathrm{GL}(d-1) \times \mathrm{SL}(2) \subset E_{d(d)}$:

$$\partial_M = \left(\partial_\alpha \,,\; \partial^{\alpha,u} \,,\; \partial^{\alpha_1\alpha_2\alpha_3} \,,\; \partial^{\alpha_1 \ldots \alpha_5,u} \,, \ldots\right) = \left(\partial_\alpha \,,\; 0 \,,\; \ldots \,,\; 0\right) \,.$$

- IIB gen geometry: $\mathcal{R}_1 \simeq TM \oplus 2 \cdot \Lambda T^*M \oplus \Lambda^3 T^*M \oplus 2 \cdot \Lambda^5 T^*M \oplus \ldots$
- Action reduces to 11-d/IIB.

Maximal gauged SUGRA $D = 11 - d$ dimensions

$M_{11} = M_D \times M_d$.

- 1 gravitational supermultiplet
- $M_{scalar} \in \frac{E_{d(d)}}{H_d}$
- $\dim(R_1)$ vector fields
- Gauge group $\subset E_{d(d)}$
- Linear constraint: $P(\theta) = 0$
- Quadratic constraints: $P(\theta^2) = 0$

Maximal SUSY and generalised parallelisability

- SUSY variations:

$$\delta_\epsilon \psi \sim \nabla \epsilon + \not{F} \epsilon \,.$$

- Fluxes generate $H_d \supset \mathrm{SO}(d)$ action on spinors.
- Maximal SUSY + fluxes ⇒ maximal set of H_d spinors!
- ⇒ "generalised parallelisability" ("gen Id structure"), well-defined nowhere-vanishing generalised vector fields ${E_A}^M \in E_{d(d)}$:

$$E_A = v_A + \omega_{(2)A} + \omega_{(5)A} + \ldots \tag{2}$$

[Lee, Strickland-Constable, Waldram]

- Generalised vielbeine:

$${E_M}^A {E_N}^B \delta_{AB} = \mathcal{M}_{MN} \,.$$

- E.g. S^4, S^5, S^7 are generalised parallelisable!

Generalised Scherk-Schwarz

[Aldazabal, Baguet, Baron, Berman, Blair, Cassani, Dibitetto, Fernández-Melgarejo, Geissbühler, Graña, Hohm, Inverso, Lee, EM, Marqués, Nunez, Perry, Petrini, Pope, Roest, Samtleben, Strickland-Constable, Thompson, Trigiante, Waldram, . . .]

- "Generalised Scherk-Schwarz truncation":

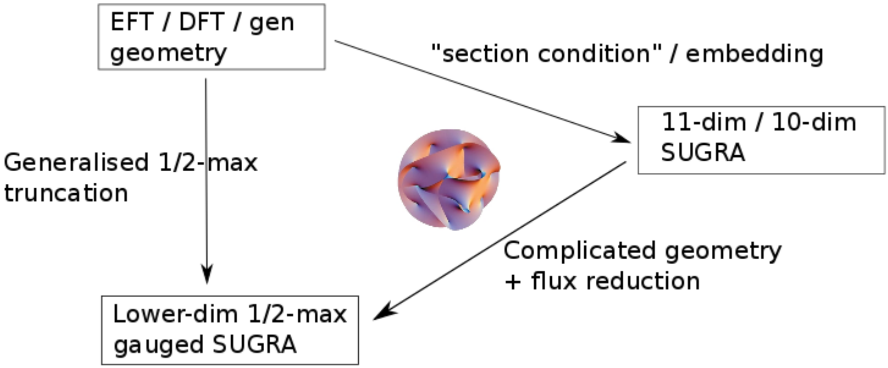
$$E_M{}^A(x, Y) = E_{\bar{M}}{}^A(x)\,(U^{-1})_M{}^{\bar{M}}(Y)\,,$$
$$A_\mu{}^M(x, Y) = A_\mu{}^{\bar{M}}(x)\,U_{\bar{M}}{}^M(Y)\,,$$

  where $U_{\bar{M}}{}^M \in E_{d(d)}$ define background Id-structure.
- Gaugings = "Intrinsic torsion"

$$\mathcal{L}_{U_{\bar{M}}} U_{\bar{N}}{}^N = \Theta_{\bar{M}\bar{N}}{}^{\bar{P}}\,U_{\bar{P}}{}^M\,.$$

  c.f. $L_{U_{\bar{m}}} U_{\bar{n}}{}^m = f_{\bar{m}\bar{n}}{}^{\bar{p}} U_{\bar{p}}{}^m$.
- $P\left(\theta\right) = 0$
- Section condition $\Rightarrow P\left(\Theta^2\right) = 0$.
- Consistency of sphere, hyperboloid truncations, Pauli truncations, non-geometric truncations . . .

How to break half the SUSY?



Half-maximal gauged SUGRA $D = 11 - d > 4$ dimensions

- 1 gravitational supermultiplet + N vector supermultiplets
- $M_{scalar} \in \frac{\mathrm{O}(d-1+N)}{\mathrm{O}(d-1)\times\mathrm{O}(N)} \times \mathbb{R}^+$
- $d - 1 + n$ vector fields
- Gauge group $\subset \mathrm{O}(d-1+N)$
- $(f_{ABC}, f_A) \oplus \begin{cases} \theta\,, & D=7\,, \\ \theta_A\,, & D=6\,, \\ \theta_{AB}\,, & D=5\,, \end{cases} \qquad A = 1, \ldots, d-1+N$
- Quadratic constraint: $f_{E[AB}f_{CD]}{}^E + \ldots = 0.$

[Bergshoeff, Gomis, Nutma, Roest]; [Schön, Weidner]

$\mathrm{G}_{\mathrm{half}}$-structures and half-maximal SUSY

- Half-maximal set of H_d spinors Ψ.
- Generalised structure group $\mathrm{G}_{\mathrm{half}} = \mathrm{SO}(d-1) \subset H_d \subset E_{d(d)}$.

D	$E_{d(d)}$	H_d	$\mathrm{G}_{\mathrm{half}}$	G_R
7	$\mathrm{SL}(5)$	$\mathrm{USp}(4)$	$\mathrm{SU}(2)$	$\mathrm{SU}(2)$
6a	$\mathrm{Spin}(5,5)$	$\mathrm{USp}(4) \times \mathrm{USp}(4)$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$
6b	$\mathrm{Spin}(5,5)$	$\mathrm{USp}(4) \times \mathrm{USp}(4)$	$\mathrm{USp}(4)$	$\mathrm{USp}(4)$
5	$E_{6(6)}$	$\mathrm{USp}(8)$	$\mathrm{USp}(4)$	$\mathrm{USp}(4)$
4	$E_{7(7)}$	$\mathrm{SU}(8)$	$\mathrm{SU}(4)$	$\mathrm{SU}(4) \times \mathrm{U}(1)$

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6b	$\mathrm{Spin}(5,5)$	$\mathrm{USp}(4) \times \mathrm{USp}(4)$	$\mathrm{USp}(4)$	$\mathrm{USp}(4)$
5	$E_{6(6)}$	$\mathrm{USp}(8)$	$\mathrm{USp}(4)$	$\mathrm{USp}(4)$
4	$E_{7(7)}$	$\mathrm{SU}(8)$	$\mathrm{SU}(4)$	$\mathrm{SU}(4) \times \mathrm{U}(1)$

- $D = 5$, $6a$, 7 similar pattern.
- $D = 4$, $6b$ slightly different.

G_{half}-structures and half-maximal SUSY

- Describe bosonically using "generalised differential forms".
- Sections of $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, ..., $\mathcal{R}_{D-3}$

D	$E_{d(d)}$	R_1	R_2	R_3	R_c
7	$\mathrm{SL}(5)$	**10**	$\overline{\mathbf{5}}$	**5**	$\emptyset$
6a	$\mathrm{Spin}(5,5)$	**16**	**10**	$\overline{\mathbf{16}}$	**1**
5	$E_{6(6)}$	**27**	$\overline{\mathbf{27}}$	**78**	**27**
4	$E_{7(7)}$	**56**	**133**	**912**	**1539**

$$\mathcal{R}_1 \xrightarrow{\wedge} \mathcal{R}_2 \xrightarrow{\wedge} \dots \xrightarrow{\wedge} \mathcal{R}_{D-3}$$

$$\mathcal{R}_{D-3} \xrightarrow{d} \dots \xrightarrow{d} \mathcal{R}_2 \xrightarrow{d} \mathcal{R}_1$$

$\mathrm{G}_{\mathrm{half}}$-structures and half-maximal SUSY

- Generalised tensors stabilised by $\mathrm{SO}(d-1) \subset E_{d(d)}$:

$$J_u{}^M \in \Gamma\left(\mathcal{R}_1\right) \,, \qquad K_I \in \Gamma\left(\mathcal{R}_2\right) \,, \qquad \hat{K}^I \in \Gamma\left(\mathcal{R}_{D-4}\right) \,,$$

$u = 1, \ldots, d-1$ of $\mathrm{SO}(d-1)_R$, satisfying

$$K \wedge \hat{K} > 0 \,, \qquad K \times_{\mathcal{R}_c} K = \hat{K} \times_{\mathcal{R}_c^*} \hat{K} = 0 \,,$$
$$J_u \wedge K = 0 \,, \qquad J_u \wedge J_v = \delta_{uv} \, K \,.$$

- Generalisations of $\omega_{(2)}$, $\Omega_{(2)}$ on K3.

$\mathrm{G}_{\mathrm{half}}$-structures and half-maximal SUSY

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- Generalisations of $\omega_{(2)}$, $\Omega_{(2)}$ on K3.
- K, $\hat{K}$ break $E_{d(d)} \longrightarrow \mathrm{SO}(d-1, d-1)$.

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$\mathrm{G}_{\mathrm{half}}$-structures and half-maximal SUSY

- Generalised tensors stabilised by $\mathrm{SO}(d-1) \subset E_{d(d)}$:

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- Generalisations of $\omega_{(2)}$, $\Omega_{(2)}$ on K3.
- K, $\hat{K}$ break $E_{d(d)} \longrightarrow \mathrm{SO}(d-1, d-1)$.
- J_u break $\mathrm{SO}(d-1, d-1) \longrightarrow \mathrm{SO}(d-1)$.
- J_u, K and $\hat{K}$ *implicitly* define generalised metric.

Example: $K3 \times T^2$, $D = 5$, $E_{6(6)}$

- vol_{K3}, Ω_U, $U = 1, \ldots, 3$,

$$\Omega_U \wedge \Omega_V = \delta_{UV} vol_{K3} .$$

- vol_{T^2}, v_A, σ_A, $A = 1, 2$

$$\imath_{v_A} \sigma_B = \delta_{AB} , \qquad \sigma_1 \wedge \sigma_2 = vol_{T^2} .$$

- $\mathrm{SO}(5)$ structure: $K \in \Gamma(\mathcal{R}_2)$, $\hat{K}, J_u \in \Gamma(\mathcal{R}_1)$.

  $\mathcal{R}_1 = TM \oplus \Lambda^2 T^* M \oplus \Lambda^5 T^* M$

  $\mathcal{R}_2 = \left(T^* M \otimes \Lambda^6 T^* M\right) \oplus \Lambda^4 T^* M \oplus T^* M$

$$K = vol_{K3} , \qquad \hat{K} = vol_{T^2} ,$$
$$J_U = \Omega_U , \qquad J_A = v_A + \sigma_A \wedge vol_{K3} ,$$

Truncation Ansatz

- Expand K, $\hat{K}$, J_u and κ in terms of "background" $\mathrm{SO}(d-1)$ structure: $n(Y), \hat{n}(Y), \omega_A(Y)$

$$n \wedge \hat{n} > 0\,, \qquad n \times_{\mathcal{R}_c} n = \hat{n} \times_{\mathcal{R}_c} \hat{n} = 0\,,$$
$$\omega_A \wedge n = 0\,, \qquad \omega_A \wedge \omega_B = \delta_{AB}\, n\,,$$

with $A = 1, \ldots, d-1$.

$$K(x,Y) = \Sigma^{-2}(x)\, n(Y)\,, \qquad \hat{K}(x,Y) = \Sigma^{2}(x)\, \hat{n}(Y)\,,$$
$$J_u(x,Y) = \Sigma^{-1}(x) b_u{}^A(x)\, \omega_A(Y)\,, \qquad A_\mu{}^M(x,Y) = a_\mu{}^A(x)\, \omega_A{}^M(Y)\,,$$

- $b_u{}^A$ and Σ are scalar fields of half-maximal gSUGRA

$$\left\{ b_u{}^A,\ \Sigma \right\} \in \mathbb{R}^+\,.$$

Truncation Ansatz

- Expand K, $\hat{K}$, J_u and κ in terms of "background" $\mathrm{SO}(d-1-N)$ structure: $n(Y), \hat{n}(Y), \omega_A(Y)$

$$n \wedge \hat{n} > 0\,, \qquad n \times_{\mathcal{R}_c} n = \hat{n} \times_{\mathcal{R}_c} \hat{n} = 0\,,$$
$$\omega_A \wedge n = 0\,, \qquad \omega_A \wedge \omega_B = \eta_{AB}\, n\,,$$

with $A = 1, \ldots, d-1+N$. $\eta_{AB} \longrightarrow \mathrm{SO}(d-1, N)$.

$$K(x,Y) = \Sigma^{-2}(x)\, n(Y)\,, \qquad \hat{K}(x,Y) = \Sigma^{2}(x)\, \hat{n}(Y)\,,$$
$$J_u(x,Y) = \Sigma^{-1}(x) b_u{}^A(x)\, \omega_A(Y)\,, \qquad A_\mu{}^M(x,Y) = a_\mu{}^A(x)\, \omega_A{}^M(Y)\,,$$

- $N \leq d-1$ vector multiplets.
- $b_u{}^A$ and Σ are scalar fields of half-maximal gSUGRA

$$\left\{ b_u{}^A,\ \Sigma \right\} \in \frac{\mathrm{SO}(d-1, N)}{\mathrm{SO}(d-1) \times \mathrm{SO}(N)} \times \mathbb{R}^+\,.$$

Consistency condition

- Consistency requires differential condition: use d and $\mathcal{L}$.
- Intrinsic torsion of $\mathrm{SO}(d-1-N)$ structure

$$
\begin{aligned}
\mathcal{L}_{\omega_A}\omega_B &= f_{AB}{}^C\omega_C + \frac{1}{2}\eta_{AB}f^C\omega_C + f_{[A}\omega_{B]} + \dots\,, \\
dn &= f^A\omega_A + \dots\,, \\
d\hat{n} &= \begin{cases} \theta n + \dots\,, & D = 7\,, \\ \theta^A\omega_A + \dots\,, & D = 6\,, \end{cases} \\
\mathcal{L}_{\hat{n}}\omega_A &= \theta_{AB}\omega^B + \dots\,, \quad \mathcal{L}_{\hat{n}}n = \dots\,, \quad D = 5\,.
\end{aligned}
$$

must satisfy

- closure condition & spinor condition
- constant condition: f_{ABC}, f_A, θ, ... are constant.

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\end{aligned}
$$

must satisfy

  - closure condition & spinor condition
  - constant condition: f_{ABC}, f_A, θ, ... are constant.
- Embedding tensor of 1/2-max gSUGRA.
- Section condition $\Rightarrow f_{E[AB}f_{CD]}{}^E + \ldots = 0$

Vacua

- From SUSY variations: 1/2-max AdS vacua

$$\mathcal{L}_{J_u} J_v = R_{uvw} J^w \,, \qquad \mathcal{L}_{J_u} \hat{K} = dK = 0 \,,$$

  and
  - $D = 7$: $d\hat{K} = \epsilon^{uvw} R_{uvw} K$,
  - $D = 6$: $d\hat{K} = \epsilon^{uvwx} R_{uvw} J_x$,
  - $D = 5$: $\mathcal{L}_{\hat{K}} J_u = \epsilon_{uvwxy} R^{vwx} J^y$ and $\mathcal{L}_{\hat{K}} K = 0$.
- 1/2-max Minkowski vacuum: $R_{uvw} = 0$
- Examples: $K3 \times T^n$, $S^4/\mathbb{Z}_k$, $S^5/\mathbb{Z}_k$, LLM, Gaiotto-Maldacena geometries, ...

1/2-maximal universal consistent truncations

Conjecture (Gauntlett & Varela)

For any SUSY solution of $D = 10$ or $D = 11$ SUGRA that consists of a warped product $AdS_D \times_w M$, there is a consistent truncation on M to a D-dimensional gauged SUGRA keeping only the gravitational supermultiplet.

- Proof in 1/2-max case in $D \geq 4$: keep only $\mathrm{G}_{\mathrm{half}}$ singlets!
- Recall ($D = 5$):

$$\mathcal{L}_{J_u} J_v = R_{uvw} J^w \,, \qquad dK = \mathcal{L}_{J_u} \hat{K} = \mathcal{L}_{\hat{K}} K = 0 \,,$$
$$\mathcal{L}_{\hat{K}} J_u = \epsilon_{uvwxy} R^{vwx} J^y \,.$$

- $J_u \longrightarrow \omega_u, \quad K \longrightarrow n, \quad \hat{K} \longrightarrow \hat{n}$
- $R_{uvw} \longrightarrow f_{uvw} \,, \quad \epsilon_{uvwxy} R^{wxy} \longrightarrow \theta_{uv} \,.$

Outlook

- New consistent truncations.
- Uplift of de Sitter gSUGRA in $D = 7$ [Dibitetto, Fernández-Melgarejo, Marqués].
- Uplift of de Roo-Wagermans angles. (related to DFT at $\mathrm{SL}(2)$ angles [Ciceri, Dibitetto, Fernandéz-Melgarejo, Guarino, Inverso]).
- New $1/2$-max Minkowski / AdS vacua?
- Effective theories / moduli.
- Related to heterotic theories (via duality); gauge enhancement.
- Other amounts of SUSY c.f. [Ashmore, Ashmore, Gabella, Graña, Petrini, Waldram], [Coimbra, Strickland-Constable].