Non-geometric fluxes and non-associativity in M-theory

Emanuel Malek

Arnold Sommerfeld Centre for Theoretical Physics,
Ludwig-Maximilians-University Munich.

Strings seminar, IPNL + ENS Lyon

C. Blair, EM arXiv:1412.0635,
M. Günaydin, D. Lüst, EM arXiv:1607.06474

Non-geometric backgrounds

- String phenomenology based on compactifications, e.g. $M_4 \times CY_3$.
- Fluxes play important role for moduli stabilisation, etc.
- T-duality suggests existence of "non-geometric fluxes" $Q_i{}^{jk}$, R^{ijk}.
- Such "T-fold" backgrounds are patched by T-dualities: not manifolds! [Hull hep-th/04061202]
- Non-geometric fluxes rich phenomenological structure.
- Related to "exotic branes" [de Boer, Shigemori arXiv:1209.6056].
- "Geometry" fails
  - Globally non-geometric background (metric, B-field globally ill-defined)
  - Locally non-geometric background (no local target space interpretation, **no point particles**, ...)
- Double field theory & exceptional field theory "geometrise" T/U-folds.

T^3 duality chain

- Consider 3-d compactification with H-flux, e.g. T^3 with H-flux.
  [Kachru, Schulz, Tripathy, Trivedy hep-th/0211182]
  [Shelton, Taylor, Wecht hep-th/0512005]
- $H_{ijk} \xrightarrow{T_i} f^i_{jk} \xrightarrow{T_j} Q^{ij}_k \xrightarrow{T_k} R^{ijk}\,, \quad (i,j,k = 1,\ldots,3).$
- T_i denotes T-duality along x^i direction.
- H_{ijk}, f^i_{jk}, Q^{ij}_k, R^{ijk} appear naturally in the embedding tensor of half-maximal gSUGRA.
- Start with T^3 with H-flux:

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\,, \qquad B_{12} = Nx^3\,. \tag{1}$$

- Identifications:
  $\left(x^1, x^2, x^3\right) \sim \left(x^1+1, x^2, x^3\right) \sim \left(x^1, x^2+1, x^3\right) \sim \left(x^1, x^2, x^3+1\right).$
- H-flux: $H_{123} = N$.

II: Twisted Torus

- Duality along x^1 gives twisted torus $\tilde{T}^3$:

$$ds^2 = \left(dx^1 - Nx^3 dx^2\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 , \qquad B_{ij} = 0 . \qquad (2)$$

- Identifications:

$$\begin{aligned} \left(x^1, x^2, x^3\right) &\sim \left(x^1 + 1, x^2, x^3\right) \sim \left(x^1, x^2 + 1, x^3\right) \\ &\sim \left(x^1 + Nx^2, x^2, x^3 + 1\right) . \end{aligned} \qquad (3)$$

- $\tilde{T}^3$ is a $U(1)$-bundle over T^2 with 1st Chern class $H_2(T^2, \mathbb{Z}) \ni [c_1] = N \neq 0$.
- Geometric flux can be defined using globally well-defined 1-forms

$$\begin{aligned} &\eta^1 = dx^1 - Nx^3 dx^2 , \qquad \eta^2 = dx^2 , \qquad \eta^3 = dx^3 , \\ &d\eta^i = f_{jk}^i \eta^j \wedge \eta^k , \qquad f_{23}^1 = N . \end{aligned} \qquad (4)$$

III: Globally non-geometric background

- Duality along x^2 gives non-geometric background.
- Buscher rules:

$$ds^2 = \frac{\left(dx^1\right)^2 + \left(dx^2\right)^2}{1 + N^2 \left(x^3\right)^2} + \left(dx^3\right)^2 , \qquad B_{12} = \frac{Nx^3}{1 + N^2 \left(x^3\right)^2} . \tag{5}$$

- Globally only defined up to T-duality transformation.
- Well-defined background in DFT with doubled coordinates $\left(x^1, x^2, x^3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\right)$.
- "Section" (polarisation) not globally well-defined: mixing of x^i and $\tilde{x}_i$.
- Globally, momentum and winding are mixed.

III: Q-flux

- DFT / generalised geometry description leads to alternative well-defined objects $\left(\hat{g}_{ij}, \beta^{ij}\right)$ [Graña, Minasian, Petrini, Waldram arXiv:0807.4527] , given by

$$\begin{aligned} \beta^{ij} &= \frac{1}{2}\left((g-B)^{-1} - (g+B)^{-1}\right)^{ij} , \\ \hat{g}_{ij} &= \frac{1}{2}\left((g-B)^{-1} + (g+B)^{-1}\right)^{-1}_{ij} . \end{aligned} \tag{6}$$

- Here they are

$$\hat{ds}^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 , \qquad \beta^{12} = Nx^3 . \tag{7}$$

- As $x^3 \to x^3 + 1$, $\beta^{12} \to \beta^{12} + N$.
- DFT / GG easily show this is a symmetry $\Rightarrow \hat{g}_{ij}$, β^{ij} well-defined.
- Background classified by "Q-flux" which is a spacetime tensor, here given by [Andriot, Hohm, Larfors, Lüst, Patalong arXiv:1202.3060]

$$Q_i^{jk} = \partial_i \beta^{jk} , \quad \Rightarrow \quad Q_3^{12} = N . \tag{8}$$

IV: R-flux

- Duality along x^3 gives locally non-geometric background.
- x^3 not isometry $\Rightarrow$ Buscher fails.
- T-duality is left/right asymmetric twist on CFT.
- Formally apply this asymmetric twist: "generalised T-duality" [Dabholkar, Hull hep-th/0512005].
- "Dual coordinates" $\tilde{x}_3$ appear.
- Winding number not conserved
- $\Rightarrow$ there are no zero-winding states: point-particle approximation fails $\Rightarrow$ no SUGRA picture.
- Target space picture non-existent.

IV: R-flux

- Background makes sense in DFT because of $\left(x^1, x^2, x^3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\right)$.
- $\Rightarrow$ Formally apply Buscher as if x^3 were isometry and exchange $x^3 \longleftrightarrow \tilde{x}_3$:

$$\hat{ds}^2 = \left(dx^2\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \,, \qquad \beta^{12} = N\tilde{x}_3 \,. \tag{9}$$

- Background classified by "R-flux" which is a spacetime tensor, here given by

$$R^{ijk} = \hat{\partial}^{[i}\beta^{jk]} \,, \quad \hat{\partial}^i = \tilde{\partial}^i + \beta^{ij}\partial_j \,, \qquad \tilde{\partial}^i = \frac{\partial}{\partial \tilde{x}_i} \,. \tag{10}$$

- We have $R^{123} = N$. [Andriot, Hohm, Larfors, Lüst, Patalong arXiv:1202.3060].

Non-geometric fluxes in M-theory

- Consider 4-d U-folds: 4-d backgrounds patched by U-dualities.
- Described naturally by $\mathrm{SL}(5)$ exceptional field theory: 10 coords x^α, $\tilde{x}_{[\alpha\beta]}$, $\alpha = 1, \ldots, 4$.
- H-flux, geometric flux and Q-flux generalise easily. But how do we generalise R^{ijk}?
- Non-geometric fluxes made uses $\hat{g}_{ij}$, β^{ij}.
- From $\mathrm{SL}(5)$ EFT, natural variables: $\left(g_{\alpha\beta}, C_{\alpha\beta\gamma}\right) \longrightarrow \left(\hat{g}_{\alpha\beta}, \Omega^{\alpha\beta\gamma}\right)$.
- $Q_\alpha^{\beta\gamma\delta} = \partial_\alpha \Omega^{\beta\gamma\delta}$.
- $R^{ijk} = \hat{\partial}^{[i}\beta^{jk]} \longrightarrow R^{\alpha\beta\gamma\delta\rho} = \tilde{\partial}^{[\alpha\beta}\Omega^{\gamma\delta\rho]} = 0$ not possible!

Local non-geometry in M-theory

- Recall: fluxes H_{ijk}, f_{ij}^{k}, Q_i^{jk}, R^{ijk} are spacetime tensors naturally appearing in embedding tensor of 1/2-maximal gSUGRA.
- M-theory non-geometric fluxes are spacetime tensors appear naturally in embedding tensor of maximal gSUGRA.
- [Blair, EM arXiv:1412.0635]$\Rightarrow$ M-theory R-flux is given by

$$R^{\alpha,\beta\gamma\delta\rho} = \hat{\partial}^{\alpha[\beta}\Omega^{\gamma\delta\rho]}\,, \tag{11}$$

with $\hat{\partial}^{\alpha\beta} = \tilde{\partial}^{\alpha\beta} + \Omega^{\alpha\beta\gamma}\partial_\gamma$ and $\tilde{\partial}^{\alpha\beta} = \frac{\partial}{\partial x_{\alpha\beta}}$.

- One can check that $R^{\alpha,\beta\gamma\delta\rho}$ transforms as a spacetime tensor.
- Toy model of non-geometric background?

M-theory non-geometry via twisted torus

- U-duality along 3 directions: 11-d background $\Leftrightarrow$ 11-d background.
- E.g. $F_{\alpha\beta\gamma\delta} \overset{U_{\beta\gamma\delta}}{\longleftrightarrow} Q_{\alpha}^{\beta\gamma\delta}$, $f_{\beta\gamma}^{\delta} \overset{U_{\alpha\beta\gamma}}{\longleftrightarrow} R^{\alpha,\beta\gamma\delta\alpha}$.
- Consider twisted torus $\tilde{T}^3 \times S^1$

$$ds_4^2 = \underbrace{\left(dx^1 - Nx^3 dx^2\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2}_{\text{twisted torus } \tilde{T}^3} + \underbrace{\left(dx^4\right)^2}_{S^1}, \qquad (12)$$

with identifications

$$\begin{aligned}\left(x^1, x^2, x^3, x^4\right) &\sim \left(x^1 + 1, x^2, x^3, x^4\right) \sim \left(x^1, x^2 + 1, x^3, x^4\right) \\ \sim \left(x^1, x^2, x^3, x^4 + 1\right) &\sim \left(x^1 + Nx^2, x^2, x^3 + 1, x^4\right) .\end{aligned} \qquad (13)$$

- Recall: $\tilde{T}^3 = U(1)$-fibration over T^2: 1st Chern class $H_2(T^2, \mathbb{Z}) \ni [c_1] = N =$ "geometric flux".

M-theory non-geometry via twisted torus

- Duality along x^2, x^3, x^4 gives R-flux background.
- Naively via Buscher: $x^3 \longleftrightarrow -\tilde{x}_{24}$ and

$$\begin{aligned} ds_{11}^2 &= \left(1+N^2\tilde{x}_{24}^2\right)^{1/3} ds_7^2 + \left(1+N^2\tilde{x}_{24}^2\right)^{1/3}\left(dx^3\right)^2 \\ &\quad + \left(1+N^2\tilde{x}_{24}^2\right)^{-2/3}\left(\left(dx^1\right)^2+\left(dx^2\right)^2+\left(dx^4\right)^2\right) , \\ C_3 &= \frac{N\tilde{x}_{24}}{1+N^2\tilde{x}_{24}^2} dx^1\wedge dx^3\wedge dx^4 \,. \end{aligned} \tag{14}$$

- "Non-geometric frame":

$$\hat{ds}_{11}^2 = ds_7^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2 \,, \quad \Omega^{134} = N\tilde{x}_{24} \,. \tag{15}$$

- R-flux: $R^{\alpha,\beta\gamma\delta\rho} = 4\hat{\partial}^{\alpha[\beta}\Omega^{\gamma\delta\rho]} \Rightarrow R^{4,1234} = N$.

Non-commutativity / non-associativity in string theory

- Non-commutativity well-studied in *open-string sector*: D-branes with non-vanishing B-field.
- Analogous to electron in magnetic field where *momenta are non-commutative*.
- What about closed string sector?
- Non-geometric fluxes lead to non-commutativity & non-associativity [Lüst arXiv:1010.1361], [Blumenhagen, Plauschinn arXiv:1010.1263], Andriot, Bakas, Condeescu, Fuchs, Florakis, Mylonas, Larfors, Patalong, Szabo, Schupp. . . .

Electrons in magnetic field

- Following Jackiw 1985: electrons in magnetic field experience non-commutativity & non-associativity appears.
- Consider electrons in $\mathbb{R}^3$ with magnetic field B: Lorentz force

$$\dot{\pi}^i = \frac{e}{2m}\epsilon^{ijk}\pi_j B_k \,, \tag{16}$$

- π^i are physical *gauge-invariant* momenta. **Not canonical momenta!**
- Energy $\Rightarrow$ Hamiltonian

$$H = \frac{\pi^i \pi_i}{2m} \,. \tag{17}$$

- Equations of motion $\Rightarrow$ twisted Poisson-bracket:

$$\left[x^i, x^j\right] = 0 \,, \qquad \left[x^i, \pi_j\right] = i\hbar\delta^i_j \,, \qquad \left[\pi_i, \pi_j\right] = ie\frac{\hbar}{c}\epsilon_{ijk}B^k \,. \tag{18}$$

- Non-commutativity of momenta.

Electrons in magnetic field

- Twisted Poisson-bracket:

$$\left[x^i, x^j\right] = 0\,, \qquad \left[x^i, \pi_j\right] = i\hbar\delta^i_j\,, \qquad [\pi_i, \pi_j] = ie\frac{\hbar}{c}\epsilon_{ijk}B^k\,. \tag{19}$$

- Non-associativity:

$$Jac(\pi_1, \pi_2, \pi_3) \equiv [\pi_1, [\pi_2, \pi_3]] + \ldots = \frac{e\hbar^2}{c}\partial_i B^i\,. \tag{20}$$

- Magnetic monopoles: $\nabla \cdot B = \rho_{mag} \neq 0 \Rightarrow$ non-associativity.
- Consider algebra of "translations":
- $U(a) = \exp(-\frac{i}{\hbar}a^i\pi_i)$ generate (non-commutative) translations.

Electrons in magnetic field

- Algebra of translations:

$$U(a_1)U(a_2) = \exp\left(-\frac{ie}{\hbar c}\Phi(a_1, a_2)\right) U(a_1 + a_2)\,,$$

$$\left(U(a_1)U(a_2)\right) U(a_3) = \exp\left(-i\frac{e}{\hbar c}\Phi(a_1, a_2, a_3)\right) U(a_1)\left(U(a_2)U(a_3)\right)\,.$$

- $\Phi(a_1, a_2)$ is flux through triangle formed by a_1, a_2, $\Phi(a_1, a_2, a_3)$ is flux through tetrahedron formed by a_1, a_2, a_3:

$$\Phi(a_1, a_2) = \int_{\Delta_{a_1\,a_2}} B \cdot dS\,, \quad \Phi(a_1, a_2, a_3) = \int_{\Delta_{a_1\,a_2\,a_3}} B \cdot dS\,, \quad (21)$$

- Algebra should be associative (QM wants linear operators on a Hilbert space).
- $\Rightarrow$ Flux quantisation: $\int \nabla \cdot B \in 2\pi \frac{\hbar c}{e}\mathbb{Z}$!

Non-geometric = non-commutative + non-associative

- CFT scattering calculations & canonical quantisation in dilute flux limit give non-commutative Q-flux algebra:

$$\left[x^i, x^j\right] = \frac{il_s^3}{\hbar} Q_k^{ij} w^k \,. \tag{22}$$

- Non-commutativity parameter $\Theta^{ij} = \frac{il_s^3}{\hbar} Q_k^{ij} w^k$ depends on winding number w^i.
- In R-flux background one finds:

$$\begin{aligned} \left[x^i, x^j\right] &= \frac{il_s^3}{\hbar} R^{ijk} p_k \,, \\ \left[x^i, p_j\right] &= i\hbar\delta^i_j \,, \qquad [p_i, p_j] = 0 \,, \\ \left[x^i, x^j, x^k\right] &= \frac{1}{3}\left[x^i, \left[x^j, x^k\right]\right] + \ldots = l_s^3 R^{ijk} \,. \end{aligned} \tag{23}$$

- Note: same algebra as for electron in magnetic field (with $x^i \leftrightarrow \pi_i$).

Non-associative strings

- $[x^i, x^j, x^k] = l_s^3 R^{ijk}$ gives "minimal volume" $\Delta x^i \Delta x^j \Delta x^k \geq l_s^3 R^{ijk} \Rightarrow$ no point-particles.
- Magnetic charge algebra and R-flux algebra are examples of *Malcev algebras*: $X \star Y = -Y \star X$, $Jac(X, Y, X \star Z) = Jac(X, Y, Z) \star X$. [Günaydin,Zumino 1985] [Bakas, Lüst arXiv:1309.3172].
- Rich mathematical framework for non-associative structures [Mylonas, Schupp, Szabo arXiv:1207.0926, arXiv:1312.1621, arXiv:1402.7306, ...] [Bakas, Lüst arXiv:1309.3172].
- Non-associative structure disappears upon momentum conservation$\Rightarrow$ crossing symmetry of CFT OPEs. [Blumenhagen, Deser, Lüst, Plauschinn, Rennecke arXiv:1106.0316].

The octonions and the R-flux algebra

- Mathematics: how is the R-flux algebra related to other non-associative structures, e.g. octonions (also a Malcev algebra)?
- Physics: how does the R-flux algebra lift to M-theory?
- **Conjecture: the answers are related**.

Octonions

- There are four division algebras: over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{O}$.
- Division algebra of real octonions $\mathbb{O}$: non-commutative, non-associative *Malcev algebra*.
- Elements: identity, I, + 7 "imaginary" units, e_A.

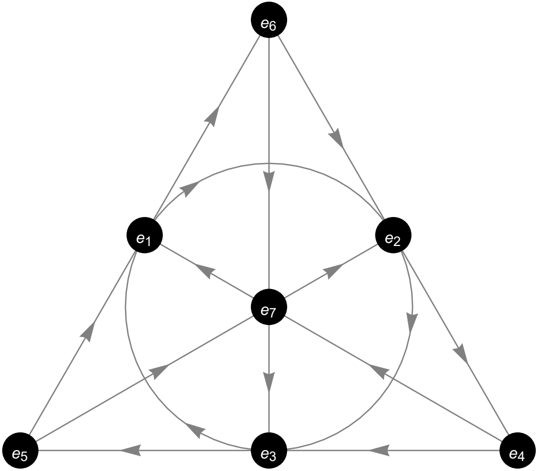
$$e_A e_B = -\delta_{AB} + \eta_{ABC} e_C \,, \qquad (A = 1, \ldots, 7) \,. \tag{24}$$

- Structure constants η_{ABC} (don't satisfy Jacobi):

$$\eta_{ABC} = 1 \Leftrightarrow (ABC) = (123)\,,(516)\,,(624)\,,(435)\,,(471)\,,(572)\,,(673)\;.$$

Fano plane of octonions

Useful mnemonic for multiplication rule: **Fano plane**



Relation with R-flux algebra

- Let $i = 1, 2, 3$ and consider e_i, $f_i = e_{i+3}$, e_7.
- Associator $[X, Y, Z] = (XY)\,Z - X\,(YZ) = \frac{1}{3}\,[X, [Y, Z]] + \ldots$.

$$
\begin{aligned}
[e_i, e_j] &= 2\epsilon_{ijk} e_k \,, \qquad [e_7, e_i] = 2f_i \,, \\
[f_i, f_j] &= -2\epsilon_{ijk} e_k \,, \qquad [e_7, f_i] = -2e_i \,, \\
[e_i, f_j] &= 2\delta_{ij} e_7 - 2\epsilon_{ijk} f_k \,, \\
[e_i, e_j, f_k] &= 4\epsilon_{ijk} e_7 - 8\delta_{k[i} f_{j]} \,, \\
[e_i, f_j, f_k] &= -8\delta_{i[j} e_{k]} \,, \\
[f_i, f_j, f_k] &= -4\epsilon_{ijk} e_7 \,, \\
[e_i, e_j, e_7] &= -4\epsilon_{ijk} f_k \,, \\
[e_i, f_j, e_7] &= -4\epsilon_{ijk} e_k \,, \\
[f_i, f_j, e_7] &= 4\epsilon_{ijk} f_k \,.
\end{aligned} \tag{25}
$$

Contraction to R-flux algebra

- Consider the contraction:

$$p_i = -\frac{1}{2} i \lambda e_i \qquad x^i = i \lambda^{1/2} \frac{\sqrt{N}}{2} f_i \qquad I = i \lambda^{3/2} \frac{\sqrt{N}}{2} e_7 \,. \qquad (26)$$

- For $\lambda \to 0$ we get

$$\begin{aligned}
[f_i, f_j] &= -2\epsilon_{ijk} e_k \Rightarrow \left[x^i, x^j\right] = iN\epsilon^{ijk} p_k \,, \\
[e_i, e_j] &= 2\epsilon_{ijk} e_k \Rightarrow [p_i, p_j] = 0 \\
[f_i, e_j] &= -\delta^i_j e_7 + \epsilon_{ijk} f_k \Rightarrow \left[x^i, p_j\right] = i\delta^i_j I \,, \\
[x_i, I] &= [p_i, I] = 0 \,, \\
[f_i, f_j, f_k] &= -4\epsilon_{ijk} e_7 \Rightarrow \left[x^i, x^j, x^k\right] = N\epsilon^{ijk} I \,.
\end{aligned} \qquad (27)$$

This gives exactly the R-flux algebra! $R^{ijk} = N\epsilon^{ijk}$

What about the uncontracted algebra?

M-theory non-associativity

- Conjecture: Uncontracted algebra is lift of R-flux algebra to M-theory.
- e_7 is 4th coordinate: $X^\alpha \sim (f_1, f_2, f_3, e_7), \quad (\alpha = 1, \dots, 4)$.
- But only three momenta $P_i \sim (e_1, e_2, e_3), \quad (i = 1, \dots, 3)$.
- Seven-dimensional phase-space.
- We can see this from duality arguments!

Homology of twisted torus

- Recall: twisted torus $\tilde{T}^3 \times S^1 \longleftrightarrow R$-flux background.
- U-duality: wrapping modes $\longleftrightarrow$ momenta.
- $H_2(\tilde{T}^3, \mathbb{Z}) = \mathbb{Z}^2 \Rightarrow$ "missing 2-cycle".
- Result follows immediately because $[c_1] \in H_2(T^2, \mathbb{Z})$ trivial in total space, $\tilde{T}^3$: $[c_1] = 0 \in H_2(\tilde{T}^3, \mathbb{Z})$.
- Or via deRham cohomology. Globally well-defined 1-forms:

$$\eta^1 = dx^1 - Nx^3 dx^2 \,, \qquad \eta^2 = dx^2 \,, \qquad \eta^3 = dx^3 \,. \tag{28}$$

- $d\eta^2 = d\eta^3 = 0$ but $d\eta^1 = f^1_{23}\eta^2 \wedge \eta^3 = N\eta^2 \wedge \eta^3 \neq 0$.
- $\Rightarrow H^1_{dR}(\tilde{T}^3) = H_2(\tilde{T}^3, \mathbb{R}) = \mathbb{R}^2$.
- $\tilde{T}^3$ is orientable $\Rightarrow H_2$ has no torsion and $H_2(\tilde{T}^3, \mathbb{Z}) = \mathbb{Z}_2$.
- "Missing 2-cycle" $\Rightarrow$ "missing momentum" after duality.

Missing momentum mode from twisted torus

- Recall: duality along x^2, x^3, x^4.
- Under duality we had

$$x^2 \longleftrightarrow \tilde{x}_{34} \,, \quad x^3 \longleftrightarrow -\tilde{x}_{24} \,, \quad x^4 \longleftrightarrow \tilde{x}_{23} \,. \tag{29}$$

- Momenta and winding modes interchanged:

$$W^{34} \longleftrightarrow P_2 \,, \qquad W^{24} \longleftrightarrow -P_1 \,, \qquad W^{23} \longleftrightarrow P_4 \,. \tag{30}$$

- But $H_2(\tilde{T}^3, \mathbb{Z}) = \mathbb{Z}^2 \Rightarrow w^{23}$ does not exist $\Rightarrow$ **no P_4 in M-theory locally non-geometric background**.
- M-theory $\leftrightarrow$ IIA duality:
  M-theory circle $X^4 \Rightarrow$ no $P_4 \Rightarrow$ no $D0$-branes in R-flux background.
- There are no point-particles (recall point-particle approximation fails)!

...from Freed-Witten anomaly cancellation condition

- Consider T^3 with H-flux: $H_3(T^3, \mathbb{Z}) \ni [H_3] \neq 0$.
- Freed-Witten anomaly: $[H_3] = 0$ on D-brane.
- $\Rightarrow$ No $D3$-branes wrapping T^3 with H-flux.
- Three T-dualities along $T^3 \Rightarrow$ No $D0$-branes on R-flux background.
- Lift to M-theory $\Rightarrow$ **no P_4 along M-theory circle!**
- **We conjecture covariant generalisation:** $R^{\alpha,\beta\gamma\delta\rho}P_\alpha = 0$. $\Rightarrow$ constraint on phase-space.

M-theory R-flux algebra

- We have $R^{4,1234} = N$ so $R^{\alpha,\beta\gamma\delta\rho} P_\alpha = 0 \Rightarrow P_4 = 0$.
- **7-dimensional phase-space**:
  coordinates X^α, momenta P_i $\quad (\alpha = 1, \ldots, 4, \quad i = 1, \ldots, 3)$.
- Identify with imaginary units of octonions

$$X^i = \frac{\sqrt{N}}{2} i l_s^{3/2} \lambda^{1/2} f_i \,, \qquad X^4 = \frac{\sqrt{N}}{2} i l_s^{3/2} \lambda^{3/2} e_7 \,, \qquad P^i = -\frac{1}{2} i\hbar\lambda e_i \,. \tag{31}$$

- **Conjecture: Octonion algebra = M-theory R-flux algebra**
- BUT: algebra cannot be $\mathrm{GL}(4)$ invariant: no 3-dim representation of $\mathrm{GL}(4)$.
- Algebra has $\mathrm{SO}(4)$ invariance: $X^\alpha \in (\mathbf{2},\mathbf{2})$, $P^i \in (\mathbf{1},\mathbf{3})$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

M-theory R-flux algebra

- **Conjecture: Octonion algebra = M-theory R-flux algebra**

$$
\begin{aligned}
[P_i, P_j] &= -i\lambda\hbar\epsilon_{ijk}P^k\,, \qquad \left[X^4, P_i\right] = i\lambda^2\hbar X_i\,,\\
\left[X^i, X^j\right] &= \frac{il_s^3}{\hbar}R^{4,ijk4}P_k\,, \qquad \left[X^4, X^i\right] = \frac{i\lambda l_s^3}{\hbar}R^{4,1234}P^i\,,\\
\left[X^i, P_j\right] &= i\hbar\delta^i_jX^4 + i\lambda\hbar\epsilon^i{}_{jk}X^k\,,\\
\left[X^\alpha, X^\beta, X^\gamma\right] &= l_s^3R^{4,\alpha\beta\gamma\delta}X_\delta\,,\\
\left[P_i, X^j, X^k\right] &= 2\lambda l_s^3R^{4,1234}\delta_i^{[j}P^{k]}\,,\\
\left[P^i, X^j, X^4\right] &= \lambda^2 l_s^3R^{4,ijk4}P_k\,,\\
[P_i, P_j, X_k] &= -\lambda^2\hbar^2\epsilon_{ijk}X^4 + 2\lambda\hbar^2\delta_{k[i}X_{j]}\,,\\
[P_i, P_j, X_4] &= \lambda^3\hbar^2\epsilon_{ijk}X_k\,,\\
[P_i, P_j, P_k] &= 0\,.
\end{aligned}
\tag{32}
$$

- $\lambda \to 0 \Rightarrow$ string R-flux algebra
- Natural identification: $\lambda \sim g_s \sim R_4/l_s$
- $\lambda \to 1$ as $g_s \to \infty$.

Summary

- M-theory non-geometric backgrounds.
- Closed-string non-associativity via non-geometry.
- Contraction of octonions gives R-flux algebra.
- M-theory R-flux background: phase-space constrained $R^{\alpha,\beta\gamma\delta\rho} P_\alpha = 0$.
- Conjecture: octonions give M-theory lift of R-flux algebra.
- Solving constraint breaks $\mathrm{GL}(4)$.

Outlook

- Can we impose constraint $R^{\alpha,\beta\gamma\delta\rho}P_\alpha = 0$ "covariantly" via *Nambu*-Dirac bracket? First-class, second-class?
- Does this explain odd-dimensional phase space?
- Derivation of $R^{\alpha,\beta\gamma\delta}P_\alpha = 0$ from duality-invariant membrane model? [Berman, Cederwall, EM arXiv:16xx.xxxx].
- "Higher"-dimensional EFT? S-dual fluxes in IIB?
- What can we learn about M-theory by requiring vanishing non-associativity?