Locally non-geometric fluxes and non-associativity in M-theory

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Based on Blair, EM arXiv:1412.0635;
Günaydin, Lüst, EM arXiv:1607.06474;
Lüst, EM, Szabo arXiv:1705.09639;
Lüst, EM, Syväri arXiv:1710.05919

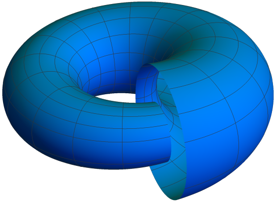
Exceptional groups & M-theory

- Hermann Nicolai, Thibault Damour: "E_{10} symmetry underlying M-theory"?
- $E_{d(d)}$, $d \leq 7$, well understood as U-duality groups
- U-duality mixes $p \longleftrightarrow$ brane wrapping modes.
- M-theory on T^d: $R_1 =$ momenta $+$ brane wrapping modes

d	$E_{d(d)}$	R_1
4	$\mathrm{SL}(5)$	$\mathbf{10} = 4 + 6$
5	$\mathrm{SO}(5,5)$	$\mathbf{16} = 5 + 10 + 1$
6	$E_{6(6)}$	$\mathbf{27} = 6 + 15 + 6$
7	$E_{7(7)}$	$\mathbf{56} = 7 + 21 + 21 + 7$

Non-geometry

- Compactification on T^6 orbifold + orientifold: [Shelton, Taylor, Wecht]
- T-duality of $D = 4$ $\mathcal{N} = 1$ superpotential $\longrightarrow$ "non-geometric fluxes": H_{ijk}, ${T_{ij}}^k$, ${Q_i}^{jk}$, R^{ijk}.
- Only H_{ijk}, ${T_{ij}}^k$ obtainable from geometric $\mathbb{R}^d/\Gamma$.
- BUT strings probe spacetime differently to point particles.
- E.g. strings on torus have winding and momentum.
- T-duality on S^1: $R \longleftrightarrow \frac{l_s^2}{R}$, $p \longleftrightarrow w$. On T^d: $O(d, d; \mathbb{Z})$.



Exceptional groups & non-geometry

- H_{ijk}, $T_{ij}{}^{k}$, $Q_i{}^{jk}$, R^{ijk}.
- $E_{d(d)}$ invariance of gaugings $\Rightarrow$ embedding tensor: geometric + several "non-geometric fluxes".
- $E_{d(d)}$ $\Rightarrow$ insight into deep "stringy" + strong-coupling regime?
- Interesting phenomenologically (moduli stabilisation, dS, ...)
- Non-associativity!

NS-NS string duality chain

- Toy model T^3 with H-flux [Kachru, Schulz, Tripathy, Trivedi]

$$H_{123} \xrightarrow{T_1} T^1{}_{23} \xrightarrow{T_2} Q^{12}{}_3 \xrightarrow{T_3} R^{123} .$$

- Generate globally non-geometric (Q-flux) and locally non-geometric (R-flux) spaces.
- H_{ijk}, $T^i{}_{jk}$, $Q^{ij}{}_k$, R^{ijk} $\subset$ embedding tensor of half-maximal gSUGRA.

I: H-flux

- Start with T^3 with H-flux:

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \,, \qquad B_{12} = Nx^3 \,.$$

- Identifications:

$$\begin{aligned} \left(x^1, x^2, x^3\right) &\sim \left(x^1 + 1, x^2, x^3\right) \sim \left(x^1, x^2 + 1, x^3\right) \\ &\sim \left(x^1, x^2, x^3 + 1\right) \end{aligned}$$

- H-flux: $H_{123} = N$.

II: Twisted Torus ($H_{123} \xrightarrow{T_1} T^1{}_{23}$)

- Duality along x^1 gives nilmanifold $\mathcal{N}_3(N) = \mathbb{R}^3/\sim$:

$$\left(x^1, x^2, x^3\right) \sim \left(x^1 + 1, x^2, x^3\right) \sim \left(x^1, x^2 + 1, x^3\right) \\ \sim \left(x^1 + N x^2, x^2, x^3 + 1\right) .$$

- Metric:

$$ds^2 = \left(dx^1 - N x^3 dx^2\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 , \qquad B_{ij} = 0 .$$

- $\mathcal{N}_3(N)$ is a principal $U(1)$-bundle over T^2 with 1st Chern class $H_2(T^2, \mathbb{Z}) \ni [c_1] = N \neq 0$.
- Geometric flux can be defined using globally well-defined 1-forms

$$e^{\bar{1}} = dx^1 - N x^3 dx^2 , \qquad e^{\bar{2}} = dx^2 , \qquad e^{\bar{3}} = dx^3 , \\ de^{\bar{i}} = T^{\bar{i}}{}_{\bar{j}\bar{k}} e^{\bar{j}} \wedge e^{\bar{k}} , \qquad T^1{}_{23} = N .$$

III: Globally non-geometric background ($T^1{}_{23} \xrightarrow{T_2} Q^{12}{}_3$)

- Duality along x^2 gives "globally non-geometric" background.
- Buscher rules:

$$ds^2 = \frac{\left(dx^1\right)^2 + \left(dx^2\right)^2}{1 + N^2 \left(x^3\right)^2} + \left(dx^3\right)^2 , \qquad B_{12} = \frac{N x^3}{1 + N^2 \left(x^3\right)^2} .$$

- Patched with T-duality as $x^3 \longrightarrow x^3 + 1$.
- Well-defined background in DFT with doubled coordinates $\left(x^1, x^2, x^3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\right)$.
- "Polarisation" ("duality frame") not globally well-defined: mixing of x^i and $\tilde{x}_i$.
- $\Rightarrow$ globally, momentum and winding are mixed.

III: Q-flux ($T^1{}_{23} \xrightarrow{T_2} Q^{12}{}_3$)

- DFT / generalised geometry description leads to alternative well-defined objects $\left(\hat{g}_{ij}, \beta^{ij}\right)$ [Graña, Minasian, Petrini, Waldram] :

$$\beta^{ij} = \frac{1}{2}\left((g-B)^{-1} - (g+B)^{-1}\right)^{ij} ,$$
$$\hat{g}_{ij} = \frac{1}{2}\left((g-B)^{-1} + (g+B)^{-1}\right)^{-1}_{ij} .$$

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- $0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0$

$$\mathcal{M}_{MN} = \begin{pmatrix} g_{ij} - B_{ik}B^j{}_k & B_i{}^j \\ -B^i{}_j & g^{ij} \end{pmatrix} ,$$

parameterisation not well-defined!

III: Q-flux $\left(T^1{}_{23} \xrightarrow{T_2} Q^{12}{}_3\right)$

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- $0 \longleftarrow T^*M \longleftarrow E \longleftarrow TM \longleftarrow 0$

$$\mathcal{M}_{MN} = \begin{pmatrix} \hat{g}_{ij} & \beta_i{}^j \\ -\beta^i{}_j & \hat{g}^{ij} - \beta^i{}_k\beta^{kj} \end{pmatrix}$$

well-defined.

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- Here they are

$$\hat{ds}^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 , \qquad \beta^{12} = Nx^3 .$$

- As $x^3 \to x^3 + 1$, $\beta^{12} \to \beta^{12} + N$. Now a symmetry!

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- As $x^3 \to x^3 + 1$, $\beta^{12} \to \beta^{12} + N$. Now a symmetry!
- Generalised metric + generalised diffeos:

$$\delta_\xi \beta^{ij} = L_\xi \beta^{ij} - 2\partial^{[i}\partial_k \xi^{j]}.$$

- Background classified by "Q-flux" which is a spacetime tensor [Andriot, Hohm, Larfors, Lüst, Patalong]:

$$Q^{ij}{}_k = \partial_k \beta^{ij} \quad \Rightarrow \quad Q^{12}{}_3 = N.$$

IV: R-flux ($Q^{12}{}_3 \xrightarrow{T_3} R^{123}$)

- Duality along x^3 gives locally non-geometric background.
- x^3 not isometry $\Rightarrow$ Buscher fails. "Generalised T-duality" [Dabholkar, Hull].
- Symmetry of background-independent string theory?
- "Dual coordinates" $\tilde{x}_3$ appear $\Rightarrow$ winding number not conserved
- $\Rightarrow$ No zero-winding states: point-particle approximation fails $\Rightarrow$ no SUGRA picture.
- No conventional "target space" picture.

IV: R-flux ($Q^{12}{}_3 \xrightarrow{T_3} R^{123}$)

- Background makes sense in DFT because of $\left(x^1, x^2, x^3, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\right)$.
- $\Rightarrow$ Formally apply Buscher as if x^3 were isometry and exchange $x^3 \longleftrightarrow \tilde{x}_3$:

$$\hat{ds}^2 = \left(dx^2\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 , \qquad \beta^{12} = N\tilde{x}_3 .$$

- Background classified by "R-flux" which is a spacetime tensor, here given by

$$R^{ijk} = \hat{\partial}^{[i}\beta^{jk]} , \quad \hat{\partial}^i = \tilde{\partial}^i + \beta^{ij}\partial_j , \qquad \tilde{\partial}^i = \frac{\partial}{\partial \tilde{x}_i} .$$

- We have $R^{123} = N$. [Andriot, Hohm, Larfors, Lüst, Patalong].

Non-commutativity / non-associativity in string theory

- Non-commutativity well-studied in *open-string sector*: D-branes with non-vanishing B-field.
- Non-associative structure arises if $H = dB \neq 0$
  e.g. [Blumenhagen, Brunner, Kupriyanov, Lüst]
- What about closed string sector?
- Non-geometric fluxes lead to non-commutativity & non-associativity
  [Lüst], [Blumenhagen, Plauschinn], Andriot, Bakas, Condeescu, Fuchs, Florakis, Mylonas, Larfors, Patalong, Szabo, Schupp. . . .

Non-geometric = non-commutative + non-associative

- CFT scattering calculations & canonical quantisation in dilute flux limit suggest non-commutative Q-flux algebra:

$$\left[x^i, x^j\right] = \frac{i l_s^3}{\hbar} Q^{ij}{}_k w^k \,.$$

- Non-commutativity parameter $\Theta^{ij} = \frac{i l_s^3}{\hbar} Q^{ij}{}_k w^k$ depends on winding number w^i.
- In R-flux background these suggest:

$$\begin{aligned} \left[x^i, x^j\right] &= \frac{i l_s^3}{\hbar} R^{ijk} p_k \,, \\ \left[x^i, p_j\right] &= i\hbar \delta^i_j \,, \qquad [p_i, p_j] = 0 \,, \\ \left[x^i, x^j, x^k\right] &= \frac{1}{3}\left[x^i, \left[x^j, x^k\right]\right] + \ldots = l_s^3 R^{ijk} \,. \end{aligned}$$

- Note: same algebra as for electron in smeared magnetic monopole field (with $x^i \leftrightarrow \pi_i$).

Non-associative strings

- $\left[x^i, x^j, x^k\right] = l_s^3 R^{ijk}$ gives "minimal volume" $\Delta x^i \Delta x^j \Delta x^k \geq l_s^3 R^{ijk} \Rightarrow$ no point-particles.
- Rich mathematical framework for non-associative structures [Mylonas, Schupp, Szabo], [Bakas, Lüst].
- Non-associative structure disappears upon momentum conservation$\Rightarrow$ crossing symmetry of CFT OPEs. [Blumenhagen, Deser, Lüst, Plauschinn, Rennecke].
- How does this generalise to strong coupling / M-theory?
- Missing momenta will appear!

Embedding tensor and non-geometric fluxes in M-theory

- Fluxes generate gaugings of lower-dim gSUGRAs.
- 4-D embedding tensor $\in \mathbf{912} \oplus \mathbf{56}$ of $\mathrm{E}_{7(7)}$.
- Under $\mathrm{GL}(7)$,

$$\mathbf{912} \oplus \mathbf{56} \longrightarrow \mathbf{1}_{-14} \oplus \mathbf{35}_{-10} \oplus \overline{\mathbf{140}}_{-6} \oplus 2 \cdot \overline{\mathbf{7}}_{-6} \oplus \mathbf{224}_{-2} \oplus 2 \cdot \mathbf{21}_{-2} \oplus \mathbf{28}_{-2} \oplus \overline{\mathbf{28}}_{2} \oplus \overline{\mathbf{21}}_{2} \oplus \overline{\mathbf{21}}_{2} \oplus \overline{\mathbf{224}}_{2} \oplus 2 \cdot \mathbf{7}_{6} \oplus \mathbf{140}_{6} \oplus \overline{\mathbf{35}}_{10} \oplus \mathbf{1}_{14} \,.$$

- Construct from

$$\mathbf{133} \longrightarrow e^{i}{}_{\bar{i}} \in \mathbf{49}_0 \,, \quad \Omega^{ijk} \in \mathbf{35}_4 \,, \quad \Omega^{ijklmn} \in \overline{\mathbf{7}}_8 \,, \quad \ldots$$
$$\partial_M \longrightarrow \partial_i \in \overline{\mathbf{7}}_{-6} \,, \quad \partial^{ij} \in \mathbf{21}_{-2} \,, \quad \partial_{ij} \in \overline{\mathbf{21}}_{2} \,, \quad \partial^{i} \in \mathbf{7}_{6} \,.$$

Locally non-geometric fluxes in M-theory

- $\mathrm{E}_{7(7)}$ generalised metric + generalised diffeos:

$$\delta_\xi \Omega^{ijk} = L_\xi \Omega^{ijk} - 3\partial^{[ij}\xi^{k]} \,,$$
$$\delta_\xi \Omega^{ijklmn} = L_\xi \Omega^{ijklmn} - \frac{1}{10}\epsilon^{ijklmnp}\partial_{pq}\xi^q - 6\Omega^{[ijk}\partial^{lm}\xi^{n]} \,.$$

- Spacetime tensors

$$R^{i,jklm} = 4\hat{\partial}^{i[j}\Omega^{klm]} - e_{\bar{i}}{}^{[j}\epsilon^{klm]inpq}\hat{\partial}_{pq}e^{\bar{i}}{}_n \,,$$
$$R^{ij}{}_k = \hat{\partial}_{kl}\Omega^{ijl} - \frac{1}{72}\epsilon_{klmnpqr}\hat{\partial}^{ij}\Omega^{lmnpqr} + \dots \,,$$
$$R^i = \hat{\partial}_{jk}\Omega^{ijk} - 4e_{\bar{i}}{}^j\hat{\partial}^i e^{\bar{i}}{}_j - 8e_{\bar{i}}{}^i\hat{\partial}^j e^{\bar{i}}{}_j \,,$$
$$R^{ijklmnp} = \hat{\partial}^{[i}\Omega^{jklmnp]} - 2\Omega^{[ijk}\hat{\partial}^m\Omega^{lnp]} \,,$$
$$R^{ijkl} = \frac{5}{8}\hat{\partial}^{[i}\Omega^{jkl]} + \frac{1}{2}\hat{\partial}_{pq}\Omega^{pqijkl} + \frac{1}{4}\Omega^{[ijk}\hat{\partial}_{pq}\Omega^{l]pq} \,.$$

M-theory toy model?

- Recall string theory toy model:

$$\underbrace{H_{123}}_{IIB} \xrightarrow{T_1} \underbrace{T^1{}_{23}}_{IIA} \xrightarrow{T_2} \underbrace{Q^{12}{}_3}_{IIB} \xrightarrow{T_3} \underbrace{R^{123}}_{IIA} .$$

- Generalise $T^1{}_{23} \xrightarrow{T_{23}} R^{123}$ to M-theory.

Twisted torus

- Generate all R-fluxes from $\mathcal{N}_3 \times T^4$ (**912**) and $\mathcal{S}_2 \times T^5$ (**56**).
- Nilmanifold $\mathcal{N}_3(N) = \mathbb{R}^3/\sim$ with

$$\left(x^1, x^2, x^3\right) \sim \left(x^1 + 1, x^2, x^3\right) \sim \left(x^1, x^2, x^3 + 1\right) \\ \sim \left(x^1 - Nx^3, x^2 + 1, x^3\right) .$$

Can choose metric

$$ds^2_{\mathcal{N}_3} = \left(dx^1 + N\, x^2\, dx^3\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 .$$

- Parallelisable $\Rightarrow$ well-defined 1-forms

$$e^{\bar{1}} = dx^1 + Nx^2 dx^3 , \qquad e^{\bar{2}} = dx^2, \qquad e^{\bar{3}} = dx^3 . \\ de^{\bar{i}} = T^{\bar{i}}{}_{\bar{j}\bar{k}} e^{\bar{j}} \wedge e^{\bar{k}} .$$

- Geometric flux: $T^1{}_{23} = N$.

Non-unimodular geometric flux

- Generate all R-fluxes from $\mathcal{N}_3 \times T^4$ (**912**) and $\mathcal{S}_2 \times T^5$ (**56**).
- $\mathcal{S}_2 = \mathbb{R}^2 / \sim$ with

$$(x_1, x_2) \sim \left(x_1 + e^{-N x_2}, x_2 \right) ,$$

- Parallelisable $\Rightarrow$ well-defined 1-forms

$$\begin{aligned} e^{\bar{1}} &= dx_1 + N x_1 dx_2 , \\ e^{\bar{2}} &= dx_2 . \\ de^{\bar{1}} &= e^{\bar{1}} \wedge e^{\bar{2}} \Rightarrow H^2_{dR}(\mathcal{S}_2) = 0 . \end{aligned}$$

- Non-compact x_2 direction: ∞-long cylinder.
- Geometric flux: $T^2{}_{12} = N$.

Duality chains

- Two kinds of U-duality acting in 7-d:
  - U-duality taken along three directions (U_3)
  - U-duality taken along six directions (U_6)
- Read off "Buscher rules" by acting on generalised metric.

Duality chains

$$\textbf{912} \quad T^1{}_{23} \xrightarrow{U_{234}} R^{4,1234} \xrightarrow{U_{156}} R^{14}{}_7\,,$$

$$\textbf{912} \quad T^1{}_{23} \xrightarrow{U_{234567}} R^{4567} \xrightarrow{U_{123}} R\,,$$

$$\textbf{56} \quad T^1{}_{12} \xrightarrow{U_{234}} Q_2{}^{234} \xrightarrow{U_{125}} R^{5,1234} = -R^{2,1345} \xrightarrow{U_{267}} R^{25}{}_5$$

$$\xrightarrow{U_{134}} R^2 = R^{26}{}_6 = R^{27}{}_7\,.$$

Missing wrapping modes

$\mathcal{N}_3 \times T^4$ has missing wrapping modes $\left(de^{\bar{i}} = T^{\bar{i}}{}_{\bar{j}\bar{k}} e^{\bar{j}} \wedge e^{\bar{k}}\right)$

- e.g. $e^{\bar{2}} \wedge e^{\bar{3}} = \frac{1}{N} de^{\bar{1}}$,
- $\mathcal{N}_3$ closed, compact, orientable $\Rightarrow w^{23} = 0$.
- Similarly $w^{23456} = w^{23457} = w^{23567} = w^{23467} = w^{14567} = 0$.
- $w^2_{KK} = w^3_{KK} = 0$ (e.g. from self-duality of $\mathcal{N}_3 \times T^4$ under U_{124}).
- Dual to Freed-Witten anomaly in IIB (T^6 with H-flux).

$\mathcal{S}_2 \times T^5$ missing wrapping states

- x^2 non-compact.
- $w^{12} = w^{12345} = w^{12346} = \ldots = w^2_{KK} = 0$.

U-duality $\Rightarrow$ missing momentum modes!

Missing momenta

$(w^{23} = w^{23456} = w^{23457} = w^{23467} = w^{23567} = w^{14567} = w^2_{KK} = w^3_{KK} = 0)$

$T_{23}{}^1 \xrightarrow{U_{234}} R^{4,1234} \xrightarrow{U_{156}} R^{14}{}_7$

- $R^{4,1234} = N$ has $p_4 = 0$.
- $R^{14}{}_7 = N$ has $p_4 = p_1 = 0$.

Missing momenta

$(w^{23} = w^{23456} = w^{23457} = w^{23467} = w^{23567} = w^{14567} = w^2_{KK} = w^3_{KK} = 0)$

$T_{23}{}^1 \xrightarrow{U_{234}} R^{4,1234} \xrightarrow{U_{156}} R^{14}{}_7$

- $R^{4,1234} = N$ has $p_4 = 0$.
- $R^{14}{}_7 = N$ has $p_4 = p_1 = 0$.

$T_{23}{}^1 \xrightarrow{U_{234567}} R^{4567} \xrightarrow{U_{123}} R^{1234567}$

- $R^{4567} = N$ has $p_4 = p_5 = p_6 = p_7 = 0$.
- $R^{1234567} = N$ has all $p_i = 0$.

Missing momenta

$(w^{23} = w^{23456} = w^{23457} = w^{23467} = w^{23567} = w^{14567} = w^2_{KK} = w^3_{KK} = 0)$

$T_{23}{}^1 \xrightarrow{U_{234}} R^{4,1234} \xrightarrow{U_{156}} R^{14}{}_7$

- $R^{4,1234} = N$ has $p_4 = 0$.
- $R^{14}{}_7 = N$ has $p_4 = p_1 = 0$.

$T_{23}{}^1 \xrightarrow{U_{234567}} R^{4567} \xrightarrow{U_{123}} R^{1234567}$

- $R^{4567} = N$ has $p_4 = p_5 = p_6 = p_7 = 0$.
- $R^{1234567} = N$ has all $p_i = 0$.

$T_{12}{}^1 \xrightarrow{U_{234}} Q_2{}^{234} \xrightarrow{U_{125}} R^{5,1234}, R^{2,1345} \xrightarrow{U_{267}} R^{25}{}_5 \xrightarrow{U_{134}} R^2, R^{26}{}_6, R^{27}{}_7$

- $R^{5,1234} = -R^{2,1345} = N$ has $p_2 = p_5 = 0$.
- $R^{25}{}_5 = N$ has $p_2 = p_5 = 0$.
- $R^2 = R^{26}{}_6 = R^{27}{}_7 = N$ has $p_1 = p_2 = p_3 = p_4 = p_5 = 0$.

Missing momenta

- These can be summarised as:

$$
\begin{aligned}
R^{i,jklm}\, p_i &= 0\,, \\
\left(R^{ij}{}_k - 2R^{[i}\delta^{j]}_k \right) p_i &= 0\,, \\
R^{ijkl}\, p_i &= 0\,, \\
R^{ijklmnp}\, p_i &= 0\,.
\end{aligned}
$$

- $R^{11,ijkl} p_{11} = 0 \longrightarrow$ IIA string theory: no $D0$-branes.
- Matches expectations from non-associativity $\Rightarrow$ minimal volume element $\Rightarrow$ no point particles.
- Can we find non-associative algebras that govern these models?

Octonions

- Focus on 4-d compactification with $R^{4,1234} \neq 0 \Rightarrow p_4 = 0$.
- Want 7-dim phase space.
- Division algebra of octonions $\mathbb{O}$: non-commutative, non-associative algebra.
- Elements: identity, I, + 7 "imaginary" units, e_A.

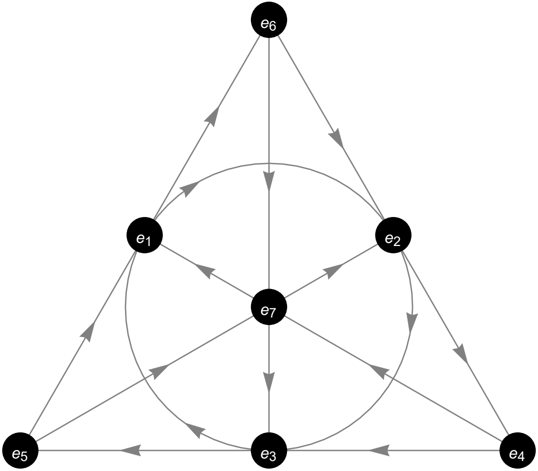
$$e_A e_B = -\delta_{AB} + \eta_{ABC} e_C \,, \qquad (A = 1, \ldots, 7) \;.$$

- Structure constants η_{ABC} (don't satisfy Jacobi):

$$\eta_{ABC} = 1 \Leftrightarrow (ABC) = (123)\,,(516)\,,(624)\,,(435)\,,(471)\,,(572)\,,(673) \;.$$

Fano plane of octonions

Useful mnemonic for multiplication rule: **Fano plane**



Relation with R-flux algebra

- Let $u = 1, 2, 3$ and consider e_u, $f_u = e_{u+3}$, e_7.
- Associator $[X, Y, Z] = (XY)\,Z - X\,(YZ) = \frac{1}{3}[X, [Y, Z]] + \ldots$.

$$
\begin{aligned}
[e_u, e_v] &= 2\epsilon_{uvw} e_w \,, \qquad [e_7, e_u] = 2f_u \,, \\
[f_u, f_v] &= -2\epsilon_{uvw} e_w \,, \qquad [e_7, f_u] = -2e_u \,, \\
[e_u, f_v] &= 2\delta_{uv} e_7 - 2\epsilon_{uvw} f_w \,, \\
[e_u, e_v, f_w] &= 4\epsilon_{uvw} e_7 - 8\delta_{w[u} f_{v]} \,, \\
[e_u, f_v, f_w] &= -8\delta_{u[v} e_{w]} \,, \\
[f_u, f_v f_w] &= -4\epsilon_{uvw} e_7 \,, \\
[e_u, e_v, e_7] &= -4\epsilon_{uvw} f_w \,, \\
[e_u, f_v, e_7] &= -4\epsilon_{uvw} e_w \,, \\
[f_u, f_v, e_7] &= 4\epsilon_{uvw} f_w \,.
\end{aligned}
$$

Contraction to R-flux algebra

- Consider the contraction:

$$p_u = -\frac{1}{2} i\lambda e_u \qquad x^u = i\lambda^{1/2}\frac{\sqrt{N}}{2} f_u \qquad I = i\lambda^{3/2}\frac{\sqrt{N}}{2} e_7 \,.$$

- For $\lambda \to 0$ we get

$$\begin{aligned}
[f_u, f_v] &= -2\epsilon_{uvw} e_k \,, \\
[e_u, e_v] &= 2\epsilon_{uvw} e_k \,, \\
[f_u, e_v] &= -\delta^u_v e_7 + \epsilon_{uvw} f_w \,, \\
[e_u, e_7] &= [f_u, e_7] = 0 \,, \\
[f_u, f_v, f_w] &= -4\epsilon_{uvw} e_7 \,.
\end{aligned}$$

Contraction to R-flux algebra

- Consider the contraction:

$$p_u = -\frac{1}{2} i\lambda e_u \qquad x^u = i\lambda^{1/2}\frac{\sqrt{N}}{2} f_u \qquad I = i\lambda^{3/2}\frac{\sqrt{N}}{2} e_7 \,.$$

- For $\lambda \to 0$ we get

$$\begin{aligned}
[x^u, x^v] &= iN\epsilon^{uvw} p_w \,, \\
[p_u, p_v] &= 0 \,, \\
[x^u, p_v] &= i\delta^u_v I \,, \\
[x_u, I] &= [p_u, I] = 0 \,, \\
[x^u, x^v, x^w] &= N\epsilon^{uvw} I \,.
\end{aligned}$$

This gives exactly the R-flux algebra! $R^{uvw} = N\epsilon^{uvw}$

What about the uncontracted algebra?

M-theory non-associativity

- Recall: $R^{4,1234} p_4 = 0 \Rightarrow p_4 = 0$.
- Conjecture: Uncontracted algebra is lift of R-flux algebra to M-theory.
- e_7 is 4th coordinate: $X^i \sim (f_1, f_2, f_3, e_7), \quad (i = 1, \ldots, 4)$.
- Three momenta $P_u \sim (e_1, e_2, e_3), \quad (i = 1, \ldots, 3)$.
- Contraction parameter $\lambda \sim g_s$ controls coupling constant.

G_2 structure and quantisation

- Octonion algebra can be deformation quantised [Kupriyanov, Szabo]
- G_2-symmetric $\star$-product realises the M-theory R-flux algebra.
- $\underbrace{\text{7-d } R\text{-flux algebra}}_{G_2}$ can be lifted to $\underbrace{\text{8-d 3-algebra}}_{\mathrm{Spin}(7)}$, including p_4.
- Constraint $p_4 = 0$ reduces 3-algebra $\longrightarrow$ M-theory R-flux algebra.

Outlook

- Non-associative algebras corresponding to $R^{i,jkl}$, $R^{ij}{}_k$, R^{ijkl}, R^i, $R^{ijklmnp}$ in lower dimensions?
- Solutions rather than toy model.
- Beyond $\mathcal{N}_3$, e.g. not parallelisable.
  Geometric flux $\Rightarrow$ 1st Chern class of $U(1)$-fibration.
- New R-branes [Bakhmatov, Berman, Kleinschmidt, Musaev, Otsuki].
- Missing momenta topological. More general derivation from ExFT? Exceptional cohomology controlling allowed momentum / winding states?

Duality chains

- Two kinds of U-duality acting in 7-d:
  - U-duality taken along three directions (U_3)
    - ⋆ M-theory $\xrightarrow{S^1\longrightarrow 0}$ IIA $\xrightarrow{S^1\longrightarrow 0}$ IIB
    - ⋆ M-theory $\xrightarrow{T^2\longrightarrow 0}$ IIB
    - ⋆ M-theory $\xrightarrow{T^3\longrightarrow 0}$ M-theory
  - U-duality taken along six directions (U_6)

M-theory R-flux algebra

- We have $R^{4,1234} = N$ so $R^{i,ijkl} P_i = 0 \Rightarrow P_4 = 0$.
- 7-dimensional phase-space:
  coordinates X^i, momenta P_u $\quad (i = 1, \ldots, 4, \quad u = 1, \ldots, 3)$.
- Identify with imaginary units of octonions

$$X^u = \frac{\sqrt{N}}{2} i l_s^{3/2} \lambda^{1/2} f_u \,, \qquad X^4 = \frac{\sqrt{N}}{2} i l_s^{3/2} \lambda^{3/2} e_7 \,, \qquad P^u = -\frac{1}{2} i \hbar \lambda e_u \,.$$

- Conjecture: Octonion algebra = M-theory R-flux algebra.

M-theory R-flux algebra

- Conjecture: Octonion algebra = M-theory R-flux algebra

$$
\begin{aligned}
[P_u, P_v] &= -i\lambda\hbar\epsilon_{uvw}P^w\,, \qquad \left[X^4, P_u\right] = i\lambda^2\hbar X_u\,, \\
[X^u, X^v] &= \frac{il_s^3}{\hbar}R^{4,uvw4}P_w\,, \qquad \left[X^4, X^u\right] = \frac{i\lambda l_s^3}{\hbar}R^{4,1234}P^u\,, \\
[X^u, P_v] &= i\hbar\delta^u_v X^4 + i\lambda\hbar\epsilon^u{}_{vw}X^w\,, \\
\left[X^i, X^j, X^k\right] &= l_s^3 R^{4,ijkl}X_l\,, \\
[P_u, X^v, X^w] &= 2\lambda l_s^3 R^{4,1234}\delta_u^{[v}P^{w]}\,, \\
\left[P^u, X^v, X^4\right] &= \lambda^2 l_s^3 R^{4,uvw4}P_w\,, \\
[P_u, P_v, X_w] &= -\lambda^2\hbar^2\epsilon_{uvw}X^4 + 2\lambda\hbar^2\delta_{w[u}X_{jv]}\,, \\
[P_u, P_v, X_4] &= \lambda^3\hbar^2\epsilon_{uvw}X_w\,, \\
[P_u, P_v, P_w] &= 0\,.
\end{aligned}
$$

- $\lambda \to 0 \Rightarrow$ string R-flux algebra
- Natural identification: $\lambda \sim g_s \sim R_4/l_s$
- $\lambda \to 1$ as $g_s \to \infty$.

Freed-Witten anomaly

- Dual IIB geometry: T^6 with H-flux, $H_{123} = N$.
- Freed-Witten: no wrapping states $w^{123} = w^{12345} = \ldots = 0$.
- M-theory $\longleftrightarrow$ IIB duality maps these to missing M-theory states.
- $\mathcal{N}_3 \times T^n$ is an F-theory description of the Freed-Witten anomaly.