Double field theory and non-geometry
Part II: M-theory and timelike dualities

Emanuel Malek

The Laboratory for Quantum Gravity & Strings,
Department of Mathematics and Applied Mathematics,
University of Cape Town

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Outline

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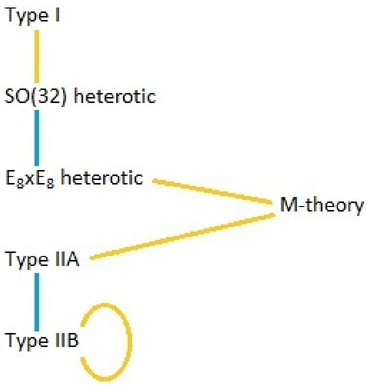
Web of dualities

String theory has perturbative T-duality and also a non-perturbative S-duality (strong-weak).



T- and S-duality relate different string theories on disparate backgrounds.

These dualities have played a key role in understanding string theory: D-branes, flux vacua, etc.
For example, the 5 superstring theories are seen as different corners of an underlying "M-theory".
Low-energy theory of M-theory is 11-D SUGRA: biggest SUGRA theory.
M-theory seen as mother of all theories.

T + S = U-duality arises when compactifying 11-D SUGRA on T^n.

n	E_n	H_n
3	$\mathrm{SL}(2) \times \mathrm{SL}(3)$	$\mathrm{SO}(2) \times \mathrm{SO}(3)$
4	$\mathrm{SL}(5)$	$\mathrm{SO}(5)$
5	$\mathrm{SO}(5,5)$	$\mathrm{SO}(5) \times \mathrm{SO}(5)$
6	E_6	$\mathrm{USp}(8)$
7	E_7	$\mathrm{SU}(8)$
8	E_8	$\mathrm{SO}(16)$

Outline

Review of DFT

Winding coordinates $\tilde{x}$ are introduced: $X^A = \begin{pmatrix} x^\mu \\ \tilde{x}_\mu \end{pmatrix}$.

Bosonic fields (of NS-NS sector) are unified in generalised metric

$$M_{AB} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho} g^{\rho\kappa} B_{\kappa\nu} & -B_{\mu\rho} g^{\rho\nu} \\ g^{\mu\rho} B_{\rho\nu} & g^{\mu\nu} \end{pmatrix} , \tag{1}$$

and generalised dilaton e^{-2d}.

Gauge transformations are unified in generalised Lie derivative

$$\mathcal{L}_U V^A = U^B \partial_B V^A - V^B \partial_B U^A + \eta^{AB} \eta_{CD} V^C \partial_B U^D . \tag{2}$$

Supergravity action $S \sim \int \left(R - H^2 + \nabla^2 \phi \right)$ is rewritten as

$$S \sim \int dx d\tilde{x} \left(\partial M \right)^2 . \tag{3}$$

Non-geometric frame

Recall that for the non-geometric string background

$$\begin{aligned} ds^2 &= \frac{1}{1+N^2x^2}\left(dy^2+dz^2\right)+dx^2\,, \\ B_{yz} &= \frac{Nx}{1+N^2x^2}\,, \end{aligned} \tag{4}$$

the generalised metric in the frame

$$M_{AB} = \begin{pmatrix} \tilde{g}_{\mu\nu} & \tilde{g}_{\mu\rho}\beta^{\rho\nu} \\ -\beta^{\mu\rho}\tilde{g}_{\rho\nu} & \tilde{g}^{\mu\nu} - \beta^{\mu\rho}\tilde{g}_{\rho\kappa}\beta^{\kappa\nu} \end{pmatrix}\,, \tag{5}$$

is more natural. The fields

$$\begin{aligned} d\tilde{s}^2 &= dx^2+dy^2+dz^2\,, \\ \beta^{yz} &= Nx\,, \end{aligned} \tag{6}$$

are periodic up to gauge transformations (the field strength ${Q^{yz}}_x$ is periodic).

Extended field theory: DFT for M-theory

How to extend this treatment to U-duality of 11-D SUGRA?
I will focus on truncation on T^4 case for simplicity: $\mathrm{SL}(5)$ U-duality group.
Consider a T^4. The M2-brane could wrap it in $\binom{4}{2} = 6$ different ways.
Therefore, include 6 wrapping coordinates $y_{\mu\nu}$.

The 10 generalised coordinates lie in the antisymmetric rep of $\mathrm{SL}(5)$, i.e. for $a, b = 1, \ldots, 5$

$$X^{[ab]} = \begin{cases} X^{\alpha 5} = x^{\alpha} \\ X^{\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta\gamma\delta} y_{\gamma\delta} \end{cases}, \tag{7}$$

where $\alpha, \beta = 1, \ldots 4$ and $\eta^{\alpha\beta\gamma\delta}$ is the 4-D alternating symbol (no $\det g$).

Generalised metric and Lie derivative

In DFT, the generalised metric and dilaton parameterised the coset space.

$$M_{AB}e^{-2d} \in \frac{\mathrm{O(D,D)}}{\mathrm{O(D)}\times\mathrm{O(D)}} \times \mathbb{R}^+ \,. \tag{8}$$

Now, we want to parameterise the coset space

$$m_{ab} \in \frac{\mathrm{SL}(5)}{\mathrm{SO}(5)} \times \mathbb{R}^+ \,. \tag{9}$$

For DFT, the generalised Lie derivative preserves the $\mathrm{O(D,D)}$ structure η_{AB}

$$\mathcal{L}_\xi \eta_{AB} = 0 \,. \tag{10}$$

Here, we need to preserve the $\mathrm{SL}(5)$ group invariant ϵ_{abcde}, the alternating tensor. We find

$$\mathcal{L}_\xi V^a = \frac{1}{2}\xi^{bc}\partial_{bc}V^a + \frac{1}{4}V^a\partial_{bc}\xi^{bc} - V^b\partial_{bc}\xi^{ac} \,. \tag{11}$$

Section condition

Closure of algebra of generalised Lie derivatives

$$[\mathcal{L}_\xi, \mathcal{L}_\chi] V^a = \mathcal{L}_{[\xi,\chi]} V^a + \text{junk} \tag{12}$$

$\Rightarrow$ "section condition" to kill junk

$$\partial_{[ab}\partial_{cd]}\Phi(X) = 0\,, \quad \partial_{[ab}\Phi(X)\partial_{cd]}\Phi'(X) = 0\,, \tag{13}$$

for all fields $\Phi(X), \Phi'(X)$ of the theory.
Conventional solution of the section condition as before comes from $4+1$ split:

$X^{\alpha 5} \equiv x^\alpha$, where $\alpha, \beta = 1, \ldots, 4$, i.e. $\partial_{\alpha 5}\Phi \neq 0$, $\partial_{\alpha\beta}\Phi = 0$.

A generalised vector

$$\xi^{ab} \rightarrow \begin{cases} \xi^{\alpha 5} = w^\alpha & \text{vector} \\ \xi^{\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta\gamma\delta}\lambda_{\gamma\delta} & \text{two-form} \end{cases}, \tag{14}$$

Will generate diffeos + 3-form gauge transformations.

The action

The action can be found by requiring invariance under generalised Lie derivatives modulo section condition.

$$S = \int_\Sigma |m|^{-1} \Big(-\frac{1}{8} m^{ab} m^{a'b'} \partial_{aa'} m^{cd} \partial_{bb'} m_{cd} + \frac{1}{2} m^{ab} m^{a'b'} \partial_{aa'} m^{cd} \partial_{cb'} m_{bd} \\ + \frac{1}{2} \partial_{aa'} m^{ab} \partial_{bb'} m^{a'b'} + \frac{3}{8} m^{ab} m^{a'b'} \partial_{aa'} \ln|m| \, \partial_{bb'} \ln|m| \\ -2 m^{a'b'} \partial_{aa'} m^{ab} \partial_{bb'} \ln|m| + m^{a'b'} \partial_{aa'} \partial_{bb'} m^{ab} - m^{ab} m^{a'b'} \partial_{aa'} \partial_{bb'} \ln|m| \Big) \tag{15}$$

where $|m| = |\det m_{ab}|$ and Σ is lower-dimensional section satisfying the section condition.

Parameterising the generalised metric

Going to $4+1$ split, 5×5 symmetric m_{ab} gives

- 4×4 symmetric $g_{\alpha\beta}$ (spacetime metric),
- vector $v^\alpha = \frac{1}{3!}\epsilon^{\alpha\beta\gamma\rho}C_{\beta\gamma\rho}$ (3-form),
- scalar ϕ (related to truncation from 11-D).

Generalised Lie derivative gives natural parameterisation

$$m_{ab} = \begin{pmatrix} \frac{g_{\alpha\beta}}{\sqrt{|g|}} & v_\alpha \\ v_\beta & \sqrt{|g|}\left(e^\phi + v^\alpha v_\alpha\right) \end{pmatrix} . \tag{16}$$

The generalised Lie derivative acts on m_{ab} as

$$\mathcal{L}_\xi m_{ab} = \frac{1}{2}\xi^{cd}\partial_{cd} m_{ab} - \frac{1}{2} m_{ab}\partial_{cd}\xi^{cd} + m_{cb}\partial_{ad}\xi^{cd} + m_{ac}\partial_{bd}\xi^{cd} . \tag{17}$$

$$\mathcal{L}_\xi m_{ab} = \frac{1}{2}\xi^{cd}\partial_{cd} m_{ab} - \frac{1}{2} m_{ab}\partial_{cd}\xi^{cd} + m_{cb}\partial_{ad}\xi^{cd} + m_{ac}\partial_{bd}\xi^{cd} \,. \quad (18)$$

Split $\xi^{ab} = \left(\xi^{\alpha 5} = w^\alpha,\ \xi^{\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta\gamma\delta}\lambda_{\gamma\delta}\right)$.
The generalised Lie derivative implies the transformations

$$\begin{aligned} \delta g_{\alpha\beta} &= L_w g_{\alpha\beta} \,, \\ \delta C_{\alpha\beta\gamma} &= L_w C_{\alpha\beta\gamma} + 3\partial_{[\alpha}\lambda_{\beta\gamma]} \,, \\ \delta\phi &= L_w \phi \,. \end{aligned} \quad (19)$$

Here L is the standard 4-D Lie derivative.
Thus w^α generates 4-D diffeomorphisms and $\lambda_{\alpha\beta}$ generates gauge transformations of the three-form potential $C_{\alpha\beta\gamma}$.

The 4-D action

Using the section, $\partial_{\alpha\beta} = 0$, and parameterisation of m_{ab}, the action becomes

$$S = -\int d^4x \; e^{2\phi} \sqrt{|g|} \left(R - \frac{1}{48} e^{-\phi} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta} + \frac{5}{2} \partial^\alpha \phi \partial_\alpha \phi \right) . \quad (20)$$

- $g_{\alpha\beta}$ – metric
- R – Ricci scalar of $g_{\alpha\beta}$.
- $F_{\alpha\beta\gamma\delta} = 4\partial_{[\alpha} C_{\beta\gamma\delta]}$ – field strength of $C_{\alpha\beta\gamma}$.

Consider truncating bosonic part of 11-dimensional supergravity to n-dimensions:

- $g_{11} = g_n \otimes e^{\varphi} \hat{g}_{11-n}$
- $|\mathrm{det}\hat{g}_{11-n}| = 1$
- cpts of 3-form only non-zero in the n directions

Integrating by parts gives

$$\sqrt{g_{11}}\, R_{11} \sim e^{(11-n)\varphi/2}\sqrt{g}\left(R_n + \frac{(11-n)(10-n)}{4} g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\right) . \quad (21)$$

so 4-D truncation is

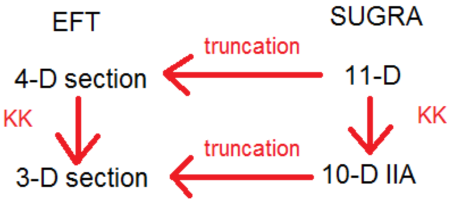
$$S = -\int d^4x\ e^{7\varphi/2}\sqrt{|h|}\left(R_h - \frac{1}{48}F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta} + \frac{21}{2}\partial^\alpha\varphi\partial_\alpha\varphi\right) . \quad (22)$$

Equivalent to previous action under conformal rescaling

$$h_{\alpha\beta} = g_{\alpha\beta}e^{-\phi/2} , \qquad u^\alpha \equiv h^{\alpha\beta}u_\beta = v^\alpha e^{\phi} , \qquad \varphi = -\frac{2}{3}\phi . \quad (23)$$

Outline

3-D truncation of IIA



Obtain a 3-D truncation of IIA by dimensional reduction of the above solution, i.e.

$$\begin{gathered}\partial_{\mu 4} = \partial_{45} = \partial_{\mu\nu} = 0\,, \\ \partial_{\mu 5} \neq 0\,,\end{gathered} \qquad \mu\,,\nu = 1\,,2\,,3\,. \tag{24}$$

- Here 4 and 5 are treated differently.
- Can we find a 3-D section where $4\,,5$ treated equally $\Rightarrow$ S-duality $\Rightarrow$ IIB?

Outline

3-D truncation of IIB

Section condition $\partial_{[ab}\partial_{cd]}\Phi = 0$ also satisfied by 3-D section $X^{\mu\nu}$, $\mu, \nu = 1, 2, 3$, i.e. $\partial_{\mu i}\Phi = \partial_{ij}\Phi = 0\,, \quad i\,, j = 4\,, 5\,.$
This treats $i\,, j = 4\,, 5$ equally and has a $\mathrm{SL}(2)$ duality.

Define

$$\tilde{x}_\mu \equiv \frac{1}{2}\eta_{\mu\nu\rho}X^{\nu\rho}\,, \qquad \tilde{\partial}^\mu \equiv \frac{1}{2}\eta^{\mu\nu\rho}\partial_{\nu\rho}\,. \tag{25}$$

$\eta_{123} = \eta^{123} = 1$ is 3-D Levi-Civita tensor *density*.

Remaining 7 coordinates related to wrapping of branes in 3-D:

IIB

- $X^{\mu\nu} \to 3$ momenta of F1,
- $X^{\mu 5} \to 3$ wrappings of F1,
- $X^{\mu 4} \to 3$ wrappings of D1,
- $X^{45} \to 1$ wrapping of D3.

IIA

- $X^{\mu 5} \to 3$ momenta of F1,
- $X^{\mu\nu} \to 3$ wrappings of F1,
- $X^{\mu 4} \to 3$ wrappings of D2,
- $X^{45} \to 1$ wrapping of D0.

Parameterising the generalised metric

Generalised Lie derivative gives different natural parameterisation

$$m_{ab} = \begin{pmatrix} \sqrt{|\tilde{g}|}\left(\tilde{g}_{\mu\nu} + e^{\tilde{\phi}}\tilde{v}^k{}_\mu \tilde{v}_{k\nu}\right) & e^{\tilde{\phi}}\tilde{v}_{j\mu} \\ e^{\tilde{\phi}}\tilde{v}_{i\nu} & \frac{1}{\sqrt{|\tilde{g}|}} e^{\tilde{\phi}}\tilde{\mathcal{M}}_{ij} \end{pmatrix} . \qquad (26)$$

5×5 symmetric m_{ab} gives

- 3×3 symmetric $\tilde{g}^{\mu\nu}$ spacetime metric, $|\tilde{g}| = |\det \tilde{g}^{\mu\nu}|$,
- 2 vectors $\tilde{v}^i{}_\mu$,
- 3 scalars: $\tilde{\phi}$, symmetric 2×2 unit-det $\tilde{\mathcal{M}}_{ij}$.

KR and RR 2-forms: $C^{i\mu\nu} \equiv \epsilon^{\mu\nu\rho}\tilde{v}^i{}_\rho$, $\tilde{\epsilon}^{123} = |\tilde{g}|^{1/2}$,

RR 0-form $\tilde{C}^{(0)}$ and string dilaton $\tilde{\varphi}$ in

$$\tilde{\mathcal{M}}_{ij} = \frac{1}{\mathrm{Im}\tau}\begin{pmatrix} |\tau|^2 & \mathrm{Re}\tau \\ \mathrm{Re}\tau & 1 \end{pmatrix} , \quad \tau = C^{(0)} + ie^{-\tilde{\varphi}} . \qquad (27)$$

Extra scalar $\tilde{\phi}$ related to truncation from 10-D.

Gauge transformations

Generalised vector ξ^{ab} contains

- 3-D vector $\tilde{\xi}_\mu \equiv \frac{1}{2}\eta_{\mu\nu\rho}\xi^{\nu\rho}$,
- $2 \times$ 3-D 1-form $\lambda^{i\mu} \equiv \xi^{i\mu}$,
- 3-D scalar ξ^{ij}.

Generalised Lie derivative

$$\mathcal{L}_\xi m_{ab} = \frac{1}{2}\xi^{cd}\partial_{cd}m_{ab} - \frac{1}{2}m_{ab}\partial_{cd}\xi^{cd} + m_{cb}\partial_{ad}\xi^{cd} + m_{ac}\partial_{bd}\xi^{cd} \,, \quad (28)$$

shows how fields transform

$$\begin{aligned} \delta\tilde{\phi} &= L_{\tilde{\xi}}\tilde{\phi}\,, & \delta\tilde{\mathcal{M}}_{ij} &= L_{\tilde{\xi}}\tilde{\mathcal{M}}_{ij}\,, \\ \delta\tilde{C}^{i\mu\nu} &= L_{\tilde{\xi}}\tilde{C}^{i\mu\nu} + 2\tilde{\partial}^{[\mu}\lambda^{|i|\nu]}\,, & \delta\tilde{g}^{\mu\nu} &= L_{\tilde{\xi}}\tilde{g}^{\mu\nu}\,. \end{aligned} \quad (29)$$

We defined

$$L_{\tilde{\xi}}V^\mu \equiv \tilde{\xi}_\nu\tilde{\partial}^\nu V^\mu + V^\nu\tilde{\partial}^\mu\tilde{\xi}_\nu\,, \quad (30)$$

NB: ξ^{ij} drops out of Lie derivative (no gauge transformation of $C^{(0)}$).

IIB action

The action $S \sim \int_\Sigma dX |m|^{-1} \ (\partial m)$ reduces to

$$S = -\int d^3\tilde{x}\sqrt{|\tilde{g}|}e^{-2\tilde{\phi}}\left(\tilde{R} + \frac{1}{4}\tilde{\partial}^\mu \tilde{\mathcal{M}}^{ij}\tilde{\partial}_\mu \tilde{\mathcal{M}}_{ij} - \frac{1}{12}e^{\tilde{\phi}}\tilde{H}^{i\mu\nu\rho}\tilde{H}_{i\mu\nu\rho} + \frac{9}{2}\tilde{\partial}^\mu\tilde{\phi}\tilde{\partial}_\mu\tilde{\phi}\right) \tag{31}$$

- μ, ν indices raised/lowered by $\tilde{g}^{\mu\nu}$,
- i, j indices are raised/lowered by $\tilde{\mathcal{M}}_{ij} \Rightarrow$ manifestly $\mathrm{SL}(2)$ invariant,
- $\mathrm{SL}(2)$ doublet of field strengths $\tilde{H}^{i\mu\nu\rho} = 3\tilde{\partial}^{[\mu}\tilde{C}^{|i|\nu\rho]}$,
- Riemann tensor $\tilde{R}$ for $\tilde{g}^{\mu\nu}$ with "reversed indices".

This is truncated IIB action with "reversed indices"!

Outline

Timelike dualities

What happens if we include time as part of the 4-D we are dualising along?

Coset space becomes $\frac{\mathrm{SL}(5)}{\mathrm{SO}(3,2)}$. (Hull, Julia arXiv/hep-th/9803239)

T-duality links IIA with IIB*, IIB with IIA*. These have RR fields with "wrong" sign for kinetic terms. (Hull arXiv/hep-th/9806146).

U-dualities change signature of spacetime: M $\rightarrow$ M* $\rightarrow$ M'. Their signatures are $(10,1) \rightarrow (9,2) \rightarrow (6,5)$. (Hull arXiv/hep-th/9807127).

Hull and Khuri hep-th/9808069 also study brane solutions of these theories (and holography in funky spacetimes hep-th/9911082).

We can understand this as follows.

Start in 11-D and compactify on a circle: we get IIA in the limit $R_1 \to 0$.

Compactify on another circle: we get IIB under T-duality in the limit $R_2 \to 0$. We have lost 2 directions ($R_1, R_2 \to 0$) but gained a T-dual direction: $\frac{1}{R_2} \to \infty$.

Compactifying on T^3 and taking $R_1, R_2, R_3 \to 0$ gives back 11-D theory.

Now include time.

Start in 11-D and compactify on a $T^{(1,1)}$. You will lose 1 spacelike & 1 timelike direction and gain 1 timelike direction. Net loss: 1 spacelike direction.

Now consider a $T^{(1,2)}$ so you go from 11-D $\to$ 11-D. This has $2 \times (1,1)$ cycles and $1 \times (0,2)$ cycle.

Thus, you lose 1 spacelike directions and gain 1 timelike direction. Signature is now $(9,2)$.

Outline

Lorentzian signature

Coset space is

$$m_{ab} \in \mathbb{R}^+ \times \frac{\mathrm{SL}(5)}{\mathrm{SO}(3,2)} \,. \tag{32}$$

m_{ab} has signature $(-, +, +, +, -)$. Choice of assigning negative directions gives different theories.
For M-theory section ($\partial_{\alpha\beta} = 0, \quad \alpha, \beta = 1, \ldots, 4$)

$$m_{ab} = \begin{pmatrix} \frac{g_{\alpha\beta}}{\sqrt{|g|}} & v_\alpha \\ v_\beta & \sqrt{|g|}\left(\pm e^\phi + v^\alpha v_\alpha\right) \end{pmatrix} \,, \tag{33}$$

with the signature of $g_{\alpha\beta}$ determining the sign of $\pm e^\phi$:

$$\begin{aligned} \mathrm{sign}(g_{\alpha\beta}) &= (-, +, +, +) \quad \& \quad - e^\phi : \quad \text{Lorentzian M-theory} \,, \\ \mathrm{sign}(g_{\alpha\beta}) &= (-, -, +, +) \quad \& e^\phi : \quad \text{M}^*\text{-theory} \,. \end{aligned} \tag{34}$$

In Einstein frame $(g_E)_{\alpha\beta} = e^{-2\phi} g_{\alpha\beta}$, **Lorentzian M-theory**

$$S = \int d^4x \sqrt{|g_E|} \left(R(g_E) - \frac{1}{48} e^{-7\phi} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta} - \frac{7}{2} \partial_\alpha \phi \, \partial^\alpha \phi \right) , \quad (35)$$

M*-theory in Einstein frame

$$S = \int d^4x \sqrt{|g_E|} \left(-R(g_E) + \frac{1}{48} e^{-7\phi} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta} + \frac{7}{2} \partial_\alpha \phi \, \partial^\alpha \phi \right) , \quad (36)$$

has *two* timelike directions.

For IIB section ($\partial_{\mu i} = \partial_{ij} = 0\,, \quad \mu, \nu = 1, 2, 3\,, \quad i, j = 4, 5$) can choose

$$\text{sign}(\tilde{g}^{\mu\nu}) = (+, -, -) \;\&\; \text{sign}(\tilde{\mathcal{M}}_{ij}) = (+, +): \quad \text{Lorentzian IIB theory}$$
$$\text{sign}(\tilde{g}^{\mu\nu}) = (-, +, +) \;\&\; \text{sign}(\tilde{\mathcal{M}}_{ij}) = (-, +): \quad \text{IIB}^* \text{ theory}$$
$$\text{sign}(\tilde{g}^{\mu\nu}) = (+, +, +) \;\&\; \text{sign}(\tilde{\mathcal{M}}_{ij}) = (-, -): \quad \text{Euclidean IIB theory}$$

Lorentzian IIB theory

$$S = \int d^3\tilde{x}\sqrt{|\tilde{g}_E|}\left(-\tilde{R}(\tilde{g}_E) - \frac{1}{4}\tilde{\partial}_\mu\tilde{\mathcal{M}}_{ij}\tilde{\partial}^\mu\tilde{\mathcal{M}}^{ij} + \frac{1}{12}e^{-7\tilde{\phi}}\tilde{H}_{i\mu\nu\rho}\tilde{H}^{i\mu\nu\rho} + \frac{7}{2}\tilde{\partial}_\mu\tilde{\phi}\,\partial^\mu\tilde{\phi}\right). \tag{37}$$

- Mostly negative spacetime signature hence $-\tilde{R}$ term.
- $\tilde{\mathcal{M}}_{ij}$ is positive definite.
- All kinetic terms have right sign.

IIB* theory

$$S=\int d^3\tilde{x}\sqrt{|\tilde{g}_E}\left(\tilde{R}(\tilde{g}_E)+\frac{1}{4}\tilde{\partial}_\mu\tilde{\mathcal{M}}_{ij}\tilde{\partial}^\mu\tilde{\mathcal{M}}^{ij}+\frac{1}{12}e^{-7\tilde{\phi}}\tilde{H}_{i\mu\nu\rho}\tilde{H}^{i\mu\nu\rho}-\frac{7}{2}\tilde{\partial}_\mu\tilde{\phi}\,\partial^\mu\tilde{\phi}\right)\,. \tag{38}$$

- Lorentzian spacetime but $\tilde{\mathcal{M}}_{ij}$ has one positive and one negative direction (parameterises $\frac{\mathrm{SL}(2)}{\mathrm{SO}(1,1)}$).
- $\tilde{\partial}^\mu\tilde{\mathcal{M}}^{ij}\tilde{\partial}_\mu\tilde{\mathcal{M}}_{ij}$ and $\tilde{\mathcal{M}}_{ij}\tilde{H}^{i\mu\nu}\tilde{H}^j{}_{\mu\nu}$ in action.
- $\Rightarrow$ One of the scalars in $\tilde{\mathcal{M}}_{ij}$ and one of the 2-forms $\tilde{C}^{i\mu\nu}$ have kinetic terms with the wrong signs.

Euclidean IIB theory

$$S = \int d^3\tilde{x}\sqrt{\tilde{g}_E}\left(-\tilde{R}(\tilde{g}_E) - \frac{1}{4}\tilde{\partial}_\mu\tilde{\mathcal{M}}_{ij}\tilde{\partial}^\mu\tilde{\mathcal{M}}^{ij} + \frac{1}{12}e^{-7\tilde{\phi}}\tilde{H}_{i\mu\nu\rho}\tilde{H}^{i\mu\nu\rho} + \frac{7}{2}\tilde{\partial}_\mu\tilde{\phi}\,\partial^\mu\tilde{\phi}\right). \tag{39}$$

Timelike dualities revisited

We found the M*, IIB* theories. Are they related by timelike dualities to M and IIB theories?

Consider a purely gravitational solution (i.e $v_\alpha = 0$)

$$m_{ab} = \begin{pmatrix} \frac{g_{\alpha\beta}}{\sqrt{|g|}} & 0 \\ 0 & -\sqrt{|g|}e^{\phi} \end{pmatrix} . \tag{40}$$

You could be forgiven for thinking that a duality can swap the $-e^{\phi}$ component for a spacelike component of $g_{\alpha\beta}$ thus giving an extra timelike direction (M*).

However, the signature of $g_{\alpha\beta}$ is fixed by η which in this case is $\eta = \mathrm{diag}\,(-1, +1, +1, +1, -1)$. Thus, $g_{\alpha\beta}$ is fixed to have signature $(3, 1)$.

What's going on?

We have assumed a parameterisation of the generalised metric,

$$m_{ab} = \begin{pmatrix} \frac{g_{\alpha\beta}}{\sqrt{|g|}} & v_\alpha \\ v_\beta & \sqrt{|g|}\left(-e^\phi + v^\alpha v_\alpha\right) \end{pmatrix} . \tag{41}$$

In terms of vielbeine, this corresponds to

$$E^{\hat{a}}{}_a = \begin{pmatrix} \frac{e^{\hat{\alpha}}{}_\alpha}{\sqrt{|e|}} & v^{\hat{\alpha}}\sqrt{|e|} \\ 0 & \sqrt{|e|}e^{\phi/2} \end{pmatrix} , \tag{42}$$

such that $m = E^T \eta E$ where $\eta = \mathrm{diag}\left(-1, +1, +1, +1, -1\right)$ in this case.

Under a duality $u \in \mathrm{SL}(5)$, the vielbein transforms as

$$E \to Eu\,. \tag{43}$$

It can also transform under the generalised Lorentz group $H = \mathrm{SO}(3,2)$, say $h(X) \in H$,

$$E \to HE. \tag{44}$$

In general, under a duality, we need to perform

$$E \to hEu\,. \tag{45}$$

Note that $h^T \eta h = \eta$ and in particular, $\mathrm{SO}(3,2)$ does not act transitively on the space of all η's.
So in general, we cannot preserve the parameterisation because this would imply changing η which is impossible.

An alternative parameterisation

An alternative parameterisation comes from the vielbein

$$E^{\hat{a}}{}_a = \begin{pmatrix} \frac{\tilde{e}^{\hat{\alpha}}{}_\alpha}{\sqrt{|\tilde{e}|}} & 0 \\ \frac{w_\alpha e^{\tilde{\phi}/2}}{\sqrt{|\tilde{e}|}} & e^{\tilde{\phi}/2}\sqrt{|\tilde{e}|} \end{pmatrix} . \tag{46}$$

The generalised metric would then have the form

$$m_{ab} = \begin{pmatrix} \frac{\tilde{g}_{\alpha\beta} - e^{\tilde{\phi}} w_\alpha w_\beta}{\sqrt{|\tilde{g}|}} & e^{\tilde{\phi}} w_\alpha \\ e^{\tilde{\phi}} w_\beta & -\sqrt{|\tilde{g}|} e^{\tilde{\phi}} \end{pmatrix} . \tag{47}$$

Thus, in the example before: swapping $-e^{\phi}$ and a spacelike component of $g_{\alpha\beta}$ would correspond to a solution in terms of this parameterisation.

What is w_α?

The parameterisation involved

$$w_\alpha = \frac{1}{3!}\epsilon_{\alpha\beta\gamma\delta}\Omega^{\beta\gamma\delta}\,, \tag{48}$$

which is a new field: a tri-vector $\Omega^{\alpha\beta\gamma}$. This is analogous to $\beta^{\alpha\beta}$ in DFT which we encountered in non-geometry. Thus, this field will also arise in non-geometry.

In general, we should consider “supergravity” with the trivector.

The EFT action $L \sim (\partial m)^2$ encodes its low-energy dynamics.

When can we use $C_{\alpha\beta\gamma}$?

We can use the standard parameterisation in terms of a metric and 3-form, only when

$$m_{\alpha\beta} \text{ has signature } (1,3)\,. \tag{49}$$

Similarly, we can only use the parameterisation with $\Omega^{\alpha\beta\gamma}$ when

$$m_{55} \geq 0\,. \tag{50}$$

Otherwise, we need to use both $C_{\alpha\beta\gamma}$ and $\Omega^{\alpha\beta\gamma}$.

In general, dualities can lead to a singular $m_{\alpha\beta}$ but this does not imply $g_{\alpha\beta}$ is singular. Instead, we should be using $\Omega^{\alpha\beta\gamma}$ parameterisation.

Outline

Summary

EFT is "bigger" than the truncation of 11-D SUGRA it came from: it also contains IIB.

In Lorentzian signature, the EFT contains M, M*, IIB, IIB* on equal footings but dualities do not relate these to each other.

Instead, timelike dualities can lead to new fields, e.g. $\Omega^{\alpha\beta\gamma}$ which also arise in non-geometries.

Future work

E_{11} and un-truncated 11-D theory?

Will higher duality groups contain more solutions to section condition?

Non-geometry, including RR non-geometry?

Construction of fully non-geometric backgrounds which cannot be dualised into geometric backgrounds.

Low and high temperature?