Advanced topics in string and field theory: Complex manifolds and Calabi-Yau manifolds

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1 Introduction

Complex manifolds are important in many different areas of string theory. For example, the Euclidean string worldsheet is a complex manifold (as we will show later on in these lectures). In this set of lectures I will focus primarily on the role of complex manifolds in the context of superstring compactifications and with this aim in mind I hope to guide you to an understanding of Calabi-Yau manifolds. Apart from Kaluza-Klein reductions, compactifying on Calabi-Yau manifolds is still one of the best understood methods of obtaining a realistic 4-dimensional low-energy theory.

In order to understand Calabi-Yau manifolds, we will meet complex manifolds, symplectic manifolds, Kähler manifolds, Betti numbers and other mathematical concepts along the way which are very important in many different areas of theoretical physics, for example gauge theories.

My goal is to give a physicists perspective on these mathematical concepts. There will be some proofs but they may not be of the rigour expected by pure mathematicians. However, for a physicist's purpose, they will suffice.

I am assuming familiarity with real differential geometry, in particular differential forms, at the level of a graduate course in General Relativity. During the first lecture, I will motivate why Calabi-Yau compactifications are useful in string theory. This will necessarily assume some knowledge of string theory and supergravity. However, beyond the motivation I will aim to keep these lectures self-contained.

1.1 Literature

There are many references available on the internet on string compactifications. In preparing these notes I found the following useful:

- "Lectures on Complex Manifolds" by Philip Candelas, (excellent thorough reference but hard to find),

- "String theory compactifications" by Mariana Graña and Hagen Triendl, (good on the low-energy theory), http://ipht.cea.fr/Docspht/articles/t13/042/public/Notes.pdf,
- "Lectures on Riemannian Geometry, Part II: Complex Manifolds" by Stefan Vandoren (nice set of notes from a physicist's perspective), http://www.staff.science.uu.nl/ vando101/MRIlectures.pdf,
- "Lectures on complex geometry, Calabi-Yau manifolds and toric geometry" by Vincent Bouchard, arXiv:hep-th/0702063 (much more advanced in certain areas),
- "Lecture Notes on Complex Differential Geometry" by Jeff Murugan.

There are also excellent textbooks which cover Calabi-Yau manifolds from a string theory perspective, for example

- Chapters 9 and 10 of "String theory and M-theory – A modern introduction" by Katrin Becker, Melanie Becker and John H. Schwarz,
- Chapters 15 and others of "Superstring theory: volume 2" by Michael B. Green, John H. Schwarz and Edward Witten.

1.2 Motivation for Calabi-Yau compactifications

String theory predicts a 10-dimensional supergravity background with $\mathcal{N}=2$ SUSY (32 supercharges), known as type II supergravity, or $\mathcal{N}=1$ SUSY (16 supercharges). The latter comes from what is known as heterotic string theory and is what we will focus on in this section.

We know that we live in a four-dimensional universe and so we are forced to look for a way to achieve this from 10-dimensional supergravity. The way to do this is to compactify six dimensions on some internal manifold so that

$$M_{10} = M_4 \times M_6 \,, \tag{1.1}$$

and we want M_4 to resemble our universe. If we take M_6 to be a T^6 we preserve all the supersymmetry and end up with a 4-dimensional $\mathcal{N}=4$ theory. While this is a very nice theory because of the large symmetry, it cannot describe our universe.

This forces us to consider a compactification which breaks some supersymmetry. We do not want to break all the supersymmetry to keep calculations simple. There are also various phenomenological reasons to expect some supersymmetry to be remnant at lower energies (for example the supersymmetric grand unification scale – 10^{16} GeV – or even the TeV scale).

Let us further assume that the four-dimensional manifold is Minkowski[1]. Note that if we have any VEVs of vector or tensor fields in M_4, these would give a preferred direction hence breaking Poincaré invariance. Such vector and tensor fields could arise from components mixing between the M_4 and M_6 as will be made clear shortly. Let us use $M, N = 1, \ldots 10$ for the 10-dimensional

[1] We will show in Exercise 1.x that we can relax this to a maximally symmetric spacetime, i.e. deSitter, Minkowski or Anti deSitter. Supersymmetry then requires the external manifold to be Minkowski.

spacetime indices, $\mu, \nu = 1, \dots 6$ for the 6-dimensional space indices (which we will mostly be concerned with here) and $m, n = 1, \dots 4$ for the 4-dimensional spacetime indices. We denote the internal coordinates by y^μ and external coordinates by x^m.

For the purpose of these lectures, all you need to know is that 10-dimensional $\mathcal{N} = 1$ supergravity contains a 3-form field strength H_{MNP}, as well as some 2-form field strengths F^i_{MN}, labelled by the index i, a scalar, known as the dilaton, and some fermions, all coupled to gravity. The fact that we do not want vector or tensor VEVs means that there is no mixing between M_4 and M_6 components of the metric and the form fields, i.e.

$$g_{\mu m} = 0\,, \qquad H_{\mu m n} = H_{\mu\nu m} = 0\,, \qquad F^i_{\mu m} = 0\,. \tag{1.2}$$

For the purpose of these lectures we make the big simplifying assumption that we have **no fluxes** turned on in the internal manifold and that the dilaton is constant. We will show that this implies the internal space is a Calabi-Yau manifold. Internal manifolds with fluxes turned on are described by generalised complex geometry and will be generalised Calabi-Yau manifolds. These are still an active area of research and we will not deal with them in this set of lectures.

To summarise, we make the following Ansatz:

- $M_{10} = M_4 \times M_6$,
- $g_{\mu m} = 0\,, \qquad H_{\mu m n} = H_{\mu\nu m} = 0\,, \qquad F^i_{\mu m} = 0$,
- No fluxes in M_6 and constant dilaton,
- M_4 is Minkowski (or maximally symmetric, see Exercise 1.1),

and we want to find M_6 so that some supersymmetry is unbroken.

Let us study the Einstein equations are

$$R_{MN} - \frac{1}{2} R g_{MN} = T_{MN}\,. \tag{1.3}$$

Here R is the Ricci scalar of R_{MN} and because of (1.2) this has a purely internal contribution $\tilde{R}$ and a four-dimensional contribution $\hat{R}$:

$$R = g^{MN} R_{MN} = g^{\mu\nu} R_{\mu\nu} + g^{mn} R_{mn} = \tilde{R} + \hat{R}\,. \tag{1.4}$$

Since the 4-dimensional manifold is Minkowski, $\hat{R} = 0$, and so we find for the internal space

$$R_{\mu\nu} - \frac{1}{2} \tilde{R} g_{\mu\nu} = T_{\mu\nu} = 0\,. \tag{1.5}$$

Since there are no fluxes present, the right hand side vanishes and we find that the internal space is Ricci-flat:

$$R_{\mu\nu} = 0\,. \tag{1.6}$$

We also want to preserve some supersymmetry. Schematically the supersymmetry equations are

$$\delta_\epsilon\,(\text{bosons}) = \epsilon\ \text{fermions}\,, \qquad \delta_\epsilon\,(\text{fermions}) = \text{bosons}\ \epsilon\,, \tag{1.7}$$

where ϵ is the (local) supersymmetry parameter. As for any classical background we want the fermions to vanish and so the bosonic variation vanishes automatically but not the variation of the fermions which instead puts a condition on our internal manifold. In fact, all the fermion variations automatically vanish in the flux-less case considered here except for the gravitino[2] variations. The gravitino variation is given by:

$$\delta\Psi_M = \nabla_M \epsilon + (\text{fluxes})_M\ \epsilon\,. \tag{1.8}$$

We need this to vanish for at least one ϵ. Thus, we need at least one spinor for which

$$\nabla_M \epsilon = 0\,. \tag{1.9}$$

This is known as the **Killing spinor equation**. If we split the spinor into an internal 6-dimensional η and external piece 4-dimensional piece ξ,

$$\epsilon(x,y) = \xi(x) \otimes \eta(y) + h.c.\,, \tag{1.10}$$

we find that $\partial_m \xi = 0$ and, more importantly,

$$\nabla_\mu \eta = 0\,. \tag{1.11}$$

The requirement of having at least one covariantly constant spinor on the internal space is a stringent requirement: it leads directly lead to Calabi-Yau manifolds, as we will now outline.[3] It is not much work (but beyond the scope of these lectures) to show that a compactification of heterotic string theory on a 6-dimensional Calabi-Yau leads to a 4-dimensional theory with $\mathcal{N} = 1$ supersymmetry.

[2]The gravitino is the superpartner of the graviton

[3]In fact, Calabi-Yau manifolds have *exactly* one covariantly constant spinor.

Exercise 1.1*: The next four exercises will show that as long as M_4 is maximally symmetric, i.e. it is Minkowski, dS or AdS, the existence of a covariantly constant spinor on M_4

$$\nabla_m \xi = 0 \,, \tag{1.12}$$

implies that M_4 is in fact Minkowski.

* *These exercises require some advanced techniques from General Relativity so if you have not seen much General Relativity yet, you may want to skip them. The remainder of these lectures does not rely on these results or the techniques used.*

Show that

$$\left[\nabla_m, \nabla_n\right] \eta = \frac{1}{2} \partial_{[m} \omega_{n]ab} \gamma^{ab} \eta + \frac{1}{4} \omega_{mab} \omega_{ncd} \left[\gamma^{ab}, \gamma^{cd}\right] \eta \,. \tag{1.13}$$

Hint: Recall that the covariant derivative of a spinor is defined as

$$\nabla_m \eta = \left(\partial_m + \frac{1}{4} \omega_{mab} \gamma^{ab} \right) \eta \,, \tag{1.14}$$

where γ^{ab} are (flat) Dirac γ-matrices and also that the Christoffel symbols $\Gamma_{mn}{}^{\rho}$ are symmetric.

Exercise 1.2*: Show that

$$\left[\gamma^{ab}, \gamma^{cd}\right] = -4 \gamma^{a[c} \eta^{d]b} + 4 \gamma^{b[c} \eta^{d]a} \,, \tag{1.15}$$

where η^{ab} is the Minkowski metric.

Hint: You may first want to show that $\gamma^{ab} = \gamma^a \gamma^b - \eta^{ab}$.

Exercise 1.3*: Use the results (1.13) and (1.15) to show that

$$\left[\nabla_m, \nabla_n\right] \eta = \frac{1}{4} R_{abmn} \gamma^{ab} \eta \,, \tag{1.16}$$

and hence

$$\nabla_m \eta = 0 \Rightarrow R_{pqmn} \gamma^{pq} \eta = 0 \,. \tag{1.17}$$

Hint: $R_{abmn} = 2 \partial_{[m} \omega_{n]ab} + 2 \omega_{[m|ac|} \omega_{n]}{}^{c}{}_{b}$.

Exercise 1.4*: Use the fact that for a maximally symmetric spacetime M_4 the Riemann tensor has the form

$$R_{mnpq} = \frac{R}{6} g_{m[p}g_{q]n} \,, \tag{1.18}$$

where R is the constant Ricci scalar of M_4 to show that (1.17) implies

$$R = 0 \,, \tag{1.19}$$

and hence that M_4 must be Minkowski.

Exercise 1.5*: Use the Bianchi identity

$$R_{[mnp]q} = 0 \,, \tag{1.20}$$

to show that for a general manifold, (1.17) implies

$$R_{mn}\gamma^m\eta = 0 \,. \tag{1.21}$$

Hint: You may first want to show that

$$\gamma^m\gamma^{np} = \gamma^{mnp} + 2g^{m[n}\gamma^{p]} \,. \tag{1.22}$$

Before we study the requirement (1.11) in more detail, we need to talk about holonomy groups.

1.2.1 Holonomy groups

For a Riemannian manifold of dimension d, the **holonomy group** describes how some objects (vectors, tensors, spinors, ...) transform under parallel transport around closed curves. Let us make this clear with some examples.

Example 1.1: The holonomy group of a general d-dimensional Riemannian manifold is $O(d)$. This is because parallel transport preserves the length of vectors, thus any vector gets mapped into a vector of equal length after parallel transport around a closed loop. The group that acts like this is $O(d)$.

Exercise 1.6: Show that the length of a vector is invariant under parallel transport.

Example 1.2: The holonomy group of a general d-dimensional orientable Riemannian manifold is $SO(d)$. This is because for orientable manifolds, vectors cannot be reflected after parallel transport around a closed loop.

Example 1.3: For a spin manifold (i.e. a manifold admitting spinors), the holonomy group is $Spin(d)$. This is because spinors transform under $Spin(d)$ (the double cover of $SO(d)$). Because $Spin(d)$ contains $SO(d)$ as a subset, we would quote $Spin(d)$ as the holonomy group of a general spin manifold.

Example 1.4: Consider first $\mathbb{R}^2$. After parallel transport around a closed loop, vectors remain unchanged. Thus, the holonomy group is trivial (it contains only the identity element).
Consider now $\mathbb{R}^2/\mathbb{Z}_4$. That is, identify all points related by a rotation by $\pi/2$. We can now form "closed loops" by taking an arc of $\pi/2$ of a circle or multiples thereof. Thus, we find that vectors will be rotated by multiples of $\pi/2$ after parallel transport around closed loops. Thus, the holonomy group is now $\mathbb{Z}_4$.

Let us now return to the compactification problem at hand. We have a spinor η that is covariantly constant and hence remains invariant under parallel transport around a closed loop. Because it is a spinor it transforms in the fundamental of $Spin(6) \simeq SU(4)$. Thus, the holonomy group must be some subset of $SU(4)$, say $H \subset SU(4)$ which keeps the spinor invariant. Note that we can use a $SU(4)$ transformation to put the spinor into the form

$$\eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta_0 \end{pmatrix} . \tag{1.23}$$

Now we can see that the subgroup H must be $SU(3)$ (or a subset thereof) which acts on the first three components of the spinor in the above form. We see that a 6-dimensional spinor remains invariant under $SU(3)$ and so this must thus be the holonomy group. We say that:

"The holonomy group is reduced to SU(3)."

These two requirements, a Ricci-flat manifold with holonomy group $SU(d/2)$, imply a **Calabi-Yau manifold**. There are various other equivalent definitions of Calabi-Yau manifolds, for example:

A Calabi-Yau manifold is a Kähler manifold with $c_1 = 0$.

The aim of these lectures is to make it clear how these two definitions are related and to study some properties of Calabi-Yau manifolds.