Advanced topics in string and field theory: Complex manifolds and Calabi-Yau manifolds

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2 Almost complex manifolds

This chapter will be devoted to understanding the consequences of the following important definition.

> **Definition:** An **almost complex manifold** (M, J) is a manifold which admits a *globally defined* rank (1,1) tensor field $J : TM \longrightarrow TM$ s.t.
>
> $$J^2 = -\mathbb{I}\,. \tag{2.1}$$
>
> A globally defined tensor satisfying (2.1) is called an **almost complex structure**.

Note: Since we are currently still talking about *real* manifolds, J is a real tensor field.

Let us look at the definition in a bit more detail. For an almost complex manifold (M, J) we have at each point $p \in M$ an endomorphism $J_p : T_pM \longrightarrow T_pM$, i.e. a map that takes vectors to vectors, which satisfies $J_p^2 = -\mathbb{I}_p$ and which depends smoothly on $p \in M$. $\mathbb{I}_p$ is here the identity matrix acting on T_pM, the tangent space at the point p. Because J is a rank $(1, 1)$ tensor we can introduce a basis of vector fields $\frac{\partial}{\partial x^\mu}$ and a dual basis of one-forms dx^μ and write at each point

$$J_p = J_\mu{}^\nu(p) \frac{\partial}{\partial x^\nu} \otimes dx^\mu\,. \tag{2.2}$$

As said earlier, J is a real-valued tensor field which means that $J_\mu{}^\nu(p)$ is real.[1] Consider a vector field $X \in \Gamma(TM)$ with components

$$X = X^\mu \partial_\mu\,. \tag{2.3}$$

J acts on this according to

$$J(X) = X^\mu J_\mu{}^\nu \partial_\nu\,, \tag{2.4}$$

[1] Caveat: $J_\mu{}^\nu$ is real when expanded in terms of real vector fields. We will see shortly that when we act on the complexified tangent vector space the components $J_\mu{}^\nu$ can be complex. However, in that case we still find that the tensor $J^\star = J$

which means that

$$J^2(X) = X^\rho J_\rho{}^\mu J_\mu{}^\nu \frac{\partial}{\partial x^\nu}\,. \tag{2.5}$$

Thus, at each point $p \in M$ we find that to be an almost complex structure J must satisfy

$$J(p)_\mu{}^\rho J(p)_\rho{}^\nu = -\delta_\mu{}^\nu\,. \tag{2.6}$$

To have an almost complex manifold, such a tensor field must be globally well-defined. This means that we must be able to define a J in any coordinate patch and in any overlap between two patches J must transform as a tensor. For a general manifold, this is not doable because of topological obstructions and we may find that, for example, J has singularities at some points.

Let us mention the simplest topological obstruction to an almost complex manifold:

Theorem 2.1: An almost complex manifold must have even dimension.

Exercise 2.1: Prove this.
Hint: Consider the determinant of J^2 and recall that J is a real-valued matrix.

Note: The converse is not true. Not all even dimensional manifolds admit an almost complex structure. For example, S^4 does not admit an almost complex structure.

Example 2.1: Let $\Sigma \in \mathbb{R}^3$ be an oriented (2-dimensional) hypersurface. Let $v : \Sigma \longrightarrow S^2$ be the Gauss map, i.e. the map which associates to every point in $p \in \Sigma$ the outer normal $v(p) \perp T_p\Sigma$. Then we can define an almost complex structure by

$$J_p u = v(p) \times u\,,\, \forall\, u \in T_p\Sigma\,, \tag{2.7}$$

where $\times$ denotes the vector cross-product.

Exercise 2.2: Show that $J_p^2 = -1$.

Theorem 2.2: Any oriented two-dimensional Riemann surface Σ is almost complex.

Proof: Σ has a metric $g_{\mu\nu}$ because it is Riemann and since it is oriented there also exists a volume-form, i.e. a covariantly constant antisymmetric tensor $\epsilon_{\mu\nu}$, which we may normalise to be $\epsilon_{12} = 1$ in a local coordinate frame. This tensor obeys

$$\epsilon^{\mu\nu}\epsilon_{\nu\rho} = -\delta^\mu{}_\rho\,, \tag{2.8}$$

where $\epsilon^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}\epsilon_{\rho\sigma}$ is raised by the metric. Now we can define the tensor field

$$J_\mu{}^\nu = \epsilon_{\mu\rho}g^{\rho\nu} \,, \tag{2.9}$$

which it can easily be verified defines an almost complex structure:

$$J^2 = -\mathbb{I} \,. \tag{2.10}$$

This completes the proof by construction.

2.1 Complexified tangent space and (anti-)holomorphic vectors

To proceed, we want to consider the **complexified tangent space**. This just means that we are now considering vectors with *complex* coefficients and linear combinations of such vectors. Addition and multiplication in these vector spaces is then just performed with complex numbers.

Concretely, we can write a general complexified vector $Z \in T_pM_{\mathbb{C}}$ as

$$Z = X + iY \,, \tag{2.11}$$

where $X, Y \in T_pM$. The complex conjugate is defined as

$$\bar{Z} = X - iY \,. \tag{2.12}$$

The reason we introduce the complexified tangent space is because we can now diagonalise J_p. J_p acts on $T_pM_{\mathbb{C}}$ as a complex linear map still satisfying

$$J_p^2 = -\mathbb{I}_p \qquad \forall\, p \in M \,. \tag{2.13}$$

The eigenvalues of J_p can only be $\pm i$ and so we have the eigenvectors in $T_pM_{\mathbb{C}}$

$$\begin{aligned} J_pZ^+ &= iZ^+ \,, \\ J_pZ^- &= -iZ^- \,. \end{aligned} \tag{2.14}$$

Exercise 2.3: Show that if $J_pZ^\pm = \pm iZ^\pm$ then

$$J_p\bar{Z}^\pm = \mp i\bar{Z}^\pm \,. \tag{2.15}$$

Hint: Write $Z^\pm$ in terms of real vectors as in (2.11) and recall that J_p is a real-valued tensor when acting on real vectors to find how it acts on X and Y.

Using the result of the above exercise, we find that J_p has equal numbers of $+i$ and $-i$ eigenvalues.

We can also define the operators

$$P^{\pm} = \frac{1}{2}\left(\mathbb{I} \mp iJ\right) , \tag{2.16}$$

satisfying

$$\left(P^{\pm}\right)^2 = P^{\pm} , \qquad P^+ + P^- = \mathbb{I} , \qquad P^+P^- = P^-P^+ = 0 . \tag{2.17}$$

These relationships mean that they are **projection operators**. Because

$$J_pP^{\pm} = \frac{1}{2}J_p\left(\mathbb{I}_p \mp iJ_p\right) = \frac{1}{2}\left(J_p \pm i\mathbb{I}_p\right) = \pm iP^{\pm} , \tag{2.18}$$

we find that these projection operators project vector fields into the $\pm i$ eigenspaces of J_p:

$$J_p\left(P^{\pm}Z\right) = \pm iP^{\pm}Z , \, \forall\, Z \in T_pM_{\mathbb{C}} . \tag{2.19}$$

Let us define the eigenspaces as

$$T_pM^{\pm} = \{Z \in T_pM_{\mathbb{C}} | J_pZ = \pm iZ\} . \tag{2.20}$$

We can write any $Z \in T_pM_{\mathbb{C}}$ as

$$Z = Z^{\pm} + Z^{\mp} , \tag{2.21}$$

where

$$Z^{\pm} \equiv P^{\pm}Z = \frac{1}{2}\left(Z \mp iJ_p(Z)\right) \tag{2.22}$$

are (anti-)holomorphic vectors. This implies that

$$T_pM_{\mathbb{C}} = T_pM^+ \oplus T_pM^- , \tag{2.23}$$

and hence if the almost complex manifold (M, J) has dimension $m = 2n$ then J_p has m eigenvalues $+i$ and m eigenvalues $-i$. We call the elements in T_pM^+ and T_pM^- **holomorphic** and **anti-holomorphic** vectors, respectively.

Let us now write two common forms of the almost complex structure. Expanding in a basis of real vectors, we can write J_p pointwise as

$$J_p = \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{pmatrix} . \tag{2.24}$$

We can also diagonalise J_p by expanding in a basis of complex vector fields in $T_pM_{\mathbb{C}}$:

$$J_p = \begin{pmatrix} i\mathbb{I}_{n\times n} & 0 \\ 0 & -i\mathbb{I}_{n\times n} \end{pmatrix} . \tag{2.25}$$

We can write this more explicitly by defining e_a and $\bar{e}_a$ to be basis vectors for $T_p M_{\mathbb{C}}$ and their corresponding dual basis e^a and $\bar{e}^a$ for the complexified cotangent space $T^*_p M_{\mathbb{C}}$, where $a = 1, \ldots n$. Then we can write the complex structure as

$$J_p = i e_a \otimes e^a - i \bar{e}_a \otimes \bar{e}^a \,, \tag{2.26}$$

where summation is implied as usual. This is often called the **canonical form** of the almost complex structure. Note that this expansion can only be done *at a point*, not even in the neighbourhood of that point! That is because the definition of the almost complex structure does not imply J must be constant.

We will soon be discussing when we can define a complex coordinate basis z^a with complex conjugates $\bar{z}^a$ such that $e_a = \frac{\partial}{\partial z^a}$ and $\bar{e}_a = \frac{\partial}{\partial \bar{z}^a}$ and similarly for the basis of one-forms $e^a = dz^a$ and $\bar{e}^a = d\bar{z}^a$. When this is possible, we have a **complex manifold**.

$\mathbb{C}$ as a vector space

In the next note we will use the fact that $\mathbb{C}$ is a vector space over $\mathbb{R}$. Let's briefly review what this means. Recall that a vector space over a field K, here $\mathbb{R}$, is a set V, here the complex numbers, subject to some axioms. The elements of V are called vectors and elements of K are called scalars. There are two operations, addition of vectors and multiplication of vectors by scalars which must satisfy the following axioms:
Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$ and a, $b \in K$, then

- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$, (associativity of vector addition)
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$, (commutativity of vector addition)
- $\exists\, \mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$, $\forall\, v \in \mathbf{V}$, (identity element of vector addition)
- $\forall\, \mathbf{v} \in V\, \exists\, {-\mathbf{v}} \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$, (inverse elements of vector addition)
- $\forall\, a\,,\, b \in K\,,\, a\,(b\mathbf{v}) = (ab)\,\mathbf{v}$, (compatibility of scalar and field multiplication)
- $1\mathbf{v} = \mathbf{v}$ where 1 is the multiplicative identity of K, (identity element of scalar multiplication)
- $a\,(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, (distributivity 1)
- $(a + b)\,\mathbf{u} = a\mathbf{u} + b\mathbf{u}$, (distributivity 2)

Before we get there, let us finish this section by discussing differential forms on almost complex manifolds.

A digression on mathematical notation

Definition: The **complexified vector space** $V_{\mathbb{C}}$ of a vector space V is defined as the tensor product

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \,. \tag{2.27}$$

This just means that an element of the complexified vector space $V_{\mathbb{C}}$ consists of a vector $v \in V$ and a complex number $c \in \mathbb{C}$ paired up as

$$(v, c) \in V_{\mathbb{C}} \,. \tag{2.28}$$

You may be worried by this because it seems like multiplication by real numbers is not well-defined. For any $r \in \mathbb{R}$, do we multiply as

$$r \cdot (v, c) = (r \cdot v, c) \ , \tag{2.29}$$

or

$$r \cdot (v, c) = (v, r \cdot c) \ ? \tag{2.30}$$

We resolve this by defining the pairing above up to an equivalence relation (as is the convention for the tensor product of any vector spaces) so that for any $r \in \mathbb{R}$

$$r\,(v, c) \sim (rv, c) \sim (v, rc) \ . \tag{2.31}$$

The fact that in the above equivalence relation $r \in \mathbb{R}$ must be real is why we labelled the tensor product with the subscript $\otimes_{\mathbb{R}}$. We can now define complex conjugation of our vectors as:

$$v \otimes c \longrightarrow v \otimes \bar{c} \,, \quad \forall\, v \in V, \ c \in \mathbb{C} \,, \tag{2.32}$$

where $\bar{c}$ is the usual complex conjugate of c.

2.2 (p, q)-forms on almost complex manifolds

In the previous section we defined operators which project vectors onto their (anti-)holomorphic parts. The projectors $P^{\pm} : TM \longrightarrow TM$ are rank (1,1) forms and hence endomorphisms of the tangent bundle. This means they also have a natural action on 1-forms. In real coordinates, the components of the projectors are real matrices $P^{\pm\nu}_{\mu}$. They thus act on a 1-form $\theta = \theta_\mu dx^\mu$ as

$$P^{\pm}\theta = P^{\pm\nu}_{\mu} \theta_\nu dx^\mu \,. \tag{2.33}$$

We define

$$\theta^{(1,0)} = P^{+}\theta\,, \qquad \theta^{(0,1)} = P^{-}\theta\,, \tag{2.34}$$

which by the properties of the projection operators satisfy

$$\theta = \theta^{(1,0)} + \theta^{(0,1)}\,, \tag{2.35}$$

and we call them $(1,0)$-forms and $(0,1)$-forms, respectively.

Exercise 2.4: Let (M, J) be an almost complex manifold and let θ be a 1-form on M. Show that

$$\theta\,(Z) = 0 \tag{2.36}$$

on any holomorphic vector field Z if and only if θ is a $(0,1)$-form.

We can also define higher (p,q)-forms. For example, for a 2-form ω we can define

$$\omega^{(2,0)}_{\mu\nu} = P^{+\rho}_{\mu} P^{+\sigma}_{\nu}\omega_{\rho\sigma}\,, \qquad \omega^{(1,1)}_{\mu\nu} = \left(P^{+\rho}_{\mu} P^{-\sigma}_{\nu} + P^{-\rho}_{\mu} P^{+\sigma}_{\nu}\right)\omega_{\rho\sigma}\,, \qquad \omega^{(2,0)}_{\mu\nu} = P^{-\rho}_{\mu} P^{-\sigma}_{\nu}\omega_{\rho\sigma}\,, \tag{2.37}$$

and these satisfy

$$\omega = \omega^{(2,0)} + \omega^{(1,1)} + \omega^{(0,2)}\,. \tag{2.38}$$

More generally let us denote the space of smooth p-forms on a manifold M as $\Omega^p(M)$ and the space of smooth p,q-form as $\Omega^{(p,q)}$. We find that this can be decomposed into a sum of lower (p,q)-forms:

$$\Omega^{p}\left(M\right) = \oplus^{p}_{k=0}\Omega^{(p-k,k)}\left(M\right)\,. \tag{2.39}$$

2.2.1 Exterior derivatives of (p,q)-forms

Let us end this chapter by discussing the exterior derivative of (p,q)-forms. Given some (p,q)-form ω, the exterior derivative acts as

$$d\omega = \left(\lambda_1\right)^{(p-1,q+2)} + \left(\lambda_2\right)^{(p,q+1)} + \left(\lambda_3\right)^{(p+1,q)} + \left(\lambda_4\right)^{(p+2,q-1)}\,, \tag{2.40}$$

where $\lambda_1, \ldots, \lambda_4$ are some $p+q+1$-forms.

Exercise 2.5: Show that this is true for the case where ω is a $(2,0)$-form and show that in this case

$$\begin{aligned}
(\lambda_1)_{\mu\nu\rho} &= 2P^{+\sigma}_{[\mu}P^{-\lambda}_{\nu}P^{-\kappa}_{\rho]}\partial_\lambda P^{+\tau}_{\kappa}\omega_{\sigma\tau}\,,\\
(\lambda_2)_{\mu\nu\rho} &= 2\omega_{\sigma\tau}P^{+\sigma}_{[\nu}\left(P^{+\lambda}_{\mu}P^{-\kappa}_{\rho]}+P^{-\lambda}_{\mu}P^{+\kappa}_{\rho]}\right)\partial_\lambda P^{+\tau}_{\kappa}+P^{-\lambda}_{[\mu}P^{+\kappa}_{\nu}P^{+\sigma}_{\rho]}\partial_\lambda\omega_{\kappa\sigma}\,,\\
(\lambda_3)_{\mu\nu\rho} &= P^{+\sigma}_{[\mu}P^{+\kappa}_{\nu}P^{+\lambda}_{\rho]}\partial_\sigma\omega_{\kappa\lambda}\,,\\
(\lambda_4)_{\mu\nu\rho} &= 0\,.
\end{aligned} \tag{2.41}$$

Hint: Use the fact that $\delta_\mu{}^\nu = P^{+\nu}_{\mu} + P^{-\nu}_{\mu}$.

The forms $(\lambda_2)^{(p,q+1)}$ and $(\lambda_3)^{(p+1,q)}$ are particularly important, as we will see in the next chapter. Let us define the **Dolbeault operators**

$$\partial : \Omega^{(p,q)} \longrightarrow \Omega^{(p+1,q)}\,, \qquad \bar{\partial} : \Omega^{(p,q)} \longrightarrow \Omega^{(p,q+1)}\,, \tag{2.42}$$

as

$$\begin{aligned}
\partial\omega^{(p,q)} &= (\lambda_3)^{(p+1,q)} = \left(P^+\right)^{p+1}\left(P^-\right)^{q} d\omega^{(p,q)}\,,\\
\bar{\partial}\omega^{(p,q+1)} &= (\lambda_2)^{(p,q+1)} = \left(P^+\right)^{p}\left(P^-\right)^{q+1} d\omega^{(p,q)}\,,
\end{aligned} \tag{2.43}$$

where we define the shorthand $(P^{\pm})^p$ as the operator that projects p indices onto their (anti-)holomorphic parts. Note that the Dolbeault operators are not nilpotent, i.e. $\partial^2 \neq 0$ and $\bar{\partial}^2 \neq 0$.

In the next chapter, we will see that for complex manifolds $d = \partial + \bar{\partial}$ and that ∂ and $\bar{\partial}$ are nilpotent.