Advanced topics in string and field theory: Complex manifolds and Calabi-Yau manifolds

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3 Complex manifolds

3.1 Complex manifolds

We will now introduce complex manifolds and discuss when an almost complex manifold is complex. Let us start by defining a complex manifold.

> **Definition:** A **complex manifold** is a differentiable manifold of even dimension $m = 2n$ with an atlas of charts to open subset of $\mathbb{C}^n$ such that the transition functions are holomorphic. The **complex dimension** of the complex manifold is then n.

That is to say, a complex manifold can be covered by open sets U_i such that each open set U_i has a coordinate map $z_i : U_i \longrightarrow C^n$ to an open subset of C^n. On each non-trivial intersection $U_i \cap U_j \neq \emptyset$, the transition functions $z_i \cdot z_j^{-1} : C^n \longrightarrow C^n$ are holomorphic. Heuristically, this definition says that in a small enough region a complex manifold "looks" (holomorphically) like $\mathbb{C}$.

The fact that the transition functions must be holomorphic is the *key property* of complex manifolds. Let us consider some examples to get a feel for the definition.

Example 3.1: The two-sphere S^2, defined by the embedding in R^3 given by

$$x^2 + y^2 + z^2 = 1\,, \tag{3.1}$$

is a complex manifold. Let us use stereographic projection from the North Pole to $\mathbb{R}^2$ as a coordinate chart (X, Y):

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)\,. \tag{3.2}$$

This covers all points on the sphere except for the North Pole itself which corresponds to $z = 1$. Thus, we introduce a second chart, which is a stereographic projection from the South Pole:

$$(U, V) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right)\,. \tag{3.3}$$

This similarly covers all points except for the South Pole ($z = -1$). In both these coordinate patches we can introduce complex coordinates (so as to map to $\mathbb{C}^1$) by defining

$$Z = X + iY\,, \qquad \bar{Z} = X - iY\,, \qquad W = U - iV\,, \qquad \bar{W} = U + iV\,. \tag{3.4}$$

On the overlap between of our two coordinate patches, i.e. $z \neq \pm 1$, we find that

$$W = \frac{1}{Z}\,. \tag{3.5}$$

This is the transition function between the coordinate patches and is *holomorphic*. Its inverse is clearly holomorphic as well. Hence S^2 is a complex manifold which can be identified with $\mathbb{C} \cup \infty$, just as a real manifold S^2 can be identified with $\mathbb{R}^2 \cup \infty$.

Example 3.2: The complex projective space $\mathbb{CP}^N$ (as explained in the note below) is a complex manifold of complex dimension N. We can see this as follows.
Let us construct an atlas on CP^N. Let z^a, $a = 1, \ldots N+1$ be complex coordinates on C^{N+1}. Then on each patch

$$U_i = \left\{ z^a,\, a = 1, \ldots, N+1 \,|\, z^i \neq 0 \right\}\,, \tag{3.6}$$

i.e. the region where $z^i \neq 0$, define the inhomogeneous coordinates

$$\xi^a_{[i]} = \frac{z^a}{z^i}\,. \tag{3.7}$$

The set of inhomogeneous coordinates $\xi^a_{[i]}$ cover the whole of CP^N since the origin of C^{N+1} is not included the definition. On the overlap of two patches $\left(U_i, \xi^a_{[i]}\right) \cap \left(U_j, \xi^a_{[j]}\right)$, we find

$$\xi^a_{[j]} = \frac{z^a}{z^k}\frac{z^k}{z^j} = \frac{\xi^a_{[k]}}{\xi^j_{[k]}}\,, \tag{3.8}$$

which is holomorphic. Hence the transition functions are holomorphic and the manifold is complex.

Complex projective spaces

The **complex projective space** will form an important example throughout these lectures. We can construct $\mathbb{CP}^N$ as a quotient of $\mathbb{C}^{N+1}$ as follows. Consider $\mathbb{C}^{N+1}\backslash\{0\}$, i.e. with the origin removed. This is characterised by $N+1$ complex numbers $z^1, \ldots, z^{N+1}$ where not all $z^a = 0$ simultaneously (here $a = 1, \ldots N+1$). To obtain $\mathbb{CP}^N$ we quotient this space by the equivalence relation

$$\left(z^1, \ldots, z^{N+1}\right) \sim \lambda \left(z^1, \ldots z^{N+1}\right) \,, \forall \lambda \in \mathbb{C} \,. \tag{3.9}$$

Any point in $\mathbb{C}^{N+1}\backslash\{0\}$ defines a (complex) line from the origin through that point and in $\mathbb{CP}^N$ we identify all points along this line. We can visualise this much better for the real projective space $\mathbb{RP}^N$ which is similarly defined as $\mathbb{R}^{N+1}\backslash\{0\}$ and with the equivalence as before but $\lambda \in \mathbb{R}$. Thus, $\mathbb{RP}^N$ really is the space of lines through the origin of $\mathbb{R}^{N+1}$.

The numbers z^a are often called **homogeneous coordinates** of $\mathbb{CP}^N$ even though they are *not* coordinates! A common set of coordinates, the so-called **inhomogeneous coordinates**, are constructed as follows: If we are in a region of $\mathbb{CP}^N$ where z^i is non-zero, we define the i-th coordinate system by rescaling the homogeneous coordinates by z^i:

$$\xi^a_{[i]} = \frac{z^a}{z^i} \,. \tag{3.10}$$

These are invariant under the equivalence relation (3.9) and so define a set of coordinates in the region where $z^i \neq 0$. Since $\xi^i_{[i]} = 1$ there are only N independent coordinates, meaning that $\mathbb{CP}^N$ is of complex dimension N. We can construct an atlas on $\mathbb{CP}^N$ from these coordinate patches by using the coordinates $\xi^a_{[i]}$ in the regions

$$U_i = \left\{ z^a = 1, \ldots, n+1 \,|\, z^i \neq 0 \right\} \,. \tag{3.11}$$

Finally, note that we can use the scaling symmetry (3.9) to set

$$|z^1|^2 + \ldots + |z^{N+1}|^2 = 1 \,, \tag{3.12}$$

which is the equation for a S^{2N+1}. This fixes the real scaling symmetry in (3.9) but not the phase rotation. Thus, we can write

$$\mathbb{CP}^N \simeq S^{2N+1}/U(1) \,. \tag{3.13}$$

Weighted projective spaces

Let us mention a generalisation of projective spaces, known as the **weigthed projective space**. We define $\mathbb{WP}^N_{[w_1,\ldots,w_{n+1}]}$ by starting as before with $\mathbb{C}^{N+1}\backslash\{0\}$ but identifying points according to the equivalence relation

$$\left(z^1,\ldots,z^{N+1}\right)\sim\left(\lambda^{w_1}z^1,\lambda^{w_2}z^2,\ldots,\lambda^{w_{N+1}}z^{N+1}\right)\,,\,\forall\,\lambda\in\mathbb{C}\,. \tag{3.14}$$

Exercise 3.1: Show that the weighted projective space $\mathbb{WP}^N$ is a complex manifold of complex dimension N.

Theorem 3.1: Complex manifolds are almost complex.

Proof: This proof relies on the crucial property that complex manifolds have holomorphic transition functions. First of all, let us consider an open neighbourhood $U\in M$ with complex coordinate chart (U,z) and define there a rank (1,1) tensor J as follows:

$$J\frac{\partial}{\partial z^a}=i\frac{\partial}{\partial z^a}\,,\qquad J\frac{\partial}{\partial\bar{z}^a}=-i\frac{\partial}{\partial\bar{z}^a}\,, \tag{3.15}$$

where z^a, $\bar{z}^a$ are the complex coordinates on U. From the definition above it is clear that $J^2=-\mathbb{I}$. The definition (3.15) is just the same as saying that in local coordinates we have the canonical form of the almost complex structure met in the previous chapter:

$$J=i\frac{\partial}{\partial z^a}\otimes dz^a-i\frac{\partial}{\partial\bar{z}^a}\otimes d\bar{z}^a\,. \tag{3.16}$$

This defines an almost complex structure in a coordinate patch. We have to now show that the almost complex structure exists globally. This is where the holomorphicity of the transition functions is crucial. Consider an overlap between two coordinate patches $(U,z)\cap(V,w)$. Because the transition functions are holomorphic we know that

$$z^a=z^a\left(w\right)\,. \tag{3.17}$$

Thus, we can calculate how the tensor J transforms between the coordinate patches. Consider

$$\frac{\partial}{\partial z^a}\otimes dz^a=\frac{\partial w^b}{\partial z^a}\frac{\partial}{\partial w^b}\otimes\frac{\partial z^a}{\partial w^c}dw^c=\frac{\partial}{\partial w^a}\otimes dw^a\,, \tag{3.18}$$

and similarly

$$\frac{\partial}{\partial\bar{z}^a}\otimes d\bar{z}^a=\frac{\partial}{\partial\bar{w}^a}\otimes d\bar{w}^a\,. \tag{3.19}$$

Thus, we see that J is globally well-defined and in particular has the same form in all coordinate patches and hence is constant throughout M. This completes the proof.

Note that we will write the components of J in terms of the complex coordinates as

$$J_a{}^b = i\,, \qquad J_{\bar{a}}{}^{\bar{b}} = -i\,, \qquad J_a{}^{\bar{b}} = J_{\bar{a}}{}^b = 0\,, \tag{3.20}$$

where

$$J = J_b{}^a \frac{\partial}{\partial z^a} \otimes dz^b + J_{\bar{b}}{}^a \frac{\partial}{\partial \bar{z}^a} \otimes dz^b + J_b{}^{\bar{a}} \frac{\partial}{\partial z^a} \otimes d\bar{z}^b + J_{\bar{b}}{}^{\bar{a}} \frac{\partial}{\partial \bar{z}^a} \otimes d\bar{z}^b\,. \tag{3.21}$$

In a complex (coordinate) basis, the matrix components $J^\mu{}_\nu$ are not necessarily real. Let us get more familiar with these with two exercises.

Exercise 3.2: Let us write a vector X in a local complex coordinate basis $X = X^\mu \frac{\partial}{\partial_\mu} = X^a \frac{\partial}{\partial z^a} + X^{\bar{a}} \frac{\partial}{\partial \bar{z}^a}$. Show that for X to be real implies

$$X^{\bar{a}} = \overline{X^a}\,. \tag{3.22}$$

Exercise 3.3: Let us write a general rank $(1,1)$ tensor J in a local complex coordinate basis as in (3.21). Show that for J to be real implies

$$J^{\bar{a}}{}_{\bar{b}} = \overline{J^a{}_b}\,, \qquad J^{\bar{a}}{}_b = \overline{J^a{}_{\bar{b}}}\,. \tag{3.23}$$

We have just shown that any complex manifold is almost complex. However, the opposite is not necessarily true. The Newlander-Nirenberg Theorem governs which almost complex manifolds are complex. Before we can understand that theorem, we first need a little bit more machinery.

3.2 Integrable complex structures

Definition: Let (M, J) be an almost complex manifold. If the Lie bracket of any two holomorphic vector fields is again a holomorphic vector field, then the almost complex structure is said to be **integrable**. An integrable almost complex structure is also called a **complex structure**.

This integrability requirement is actua.lly best expressed in terms of the Nijenhuis tensor

Definition: Let (M, J) be an almost complex manifold. Then for any two vector fields $X, Y \in \Gamma(TM)$ we define the **Nijenhuis tensor** $N : TM \otimes TM \longrightarrow TM$ as

$$N(X,Y) = [X,Y] + J\,[JX,Y] + J\,[X,JY] - [JX,JY]\,, \tag{3.24}$$

where $[X,Y]$ denotes the usual Lie bracket on M.

Exercise 3.4: Show that the Nijenhuis tensor can be written in local coordinate as

$$N_{\mu\nu}{}^{\rho} = (\partial_\mu J_\nu{}^\sigma)\, J_\sigma{}^\rho - J_\mu^\sigma\, (\partial_\sigma J_\nu{}^\rho) - (\partial_\nu J_\mu{}^\sigma)\, J_\sigma{}^\rho + J_\nu^\sigma\, (\partial_\sigma J_\mu{}^\rho)\ . \tag{3.25}$$

The definition (3.24) is written in terms of tensors and hence is manifestly coordinate invariant. The local coordinate expression (3.25), however, has lost that manifest diffeomorphism invariance. Let us restore this be rewriting the expression in terms of covariant derivatives.

Exercise 3.5: Rewrite (3.25) as

$$N_{\mu\nu}{}^{\rho} = 2J_\mu{}^\sigma \partial_{[\nu} J_{\sigma]}{}^\rho - 2J_\nu{}^\sigma \partial_{[\mu} J_{\sigma]}{}^\rho\,, \tag{3.26}$$

and hence show that it can be written in terms of covariant derivatives

$$N_{\mu\nu}{}^{\rho} = 2J_\mu{}^\sigma \nabla_{[\nu} J_{\sigma]}{}^\rho - 2J_\nu{}^\sigma \nabla_{[\mu} J_{\sigma]}{}^\rho\,. \tag{3.27}$$

Theorem 3.2: Let (M, J) be an almost complex manifold. The almost complex structure J is integrable if and only if $N(X,Y) = 0$ for any two vector fields X, $Y \in \Gamma(TM)$.

Proof: Firstly, let us extended the Nijenhuis to the complexified tangent bundle $TM_{\mathbb{C}}$. Let us start by proving the "if" part of the theorem. Since $N(X,Y) = 0$ for any two vector fields on M, we can consider X and Y to be holomorphic vectors. Thus, we have

$$N(X,Y) = [X,Y] + iJ\,[X,Y] + iJ\,[X,Y] + [X,Y] = 2\,([X,Y] + iJ\,[X,Y]) = 0\,. \tag{3.28}$$

This implies that

$$J\,[X,Y] = i\,[X,Y]\ , \tag{3.29}$$

and hence that $[X,Y]$ is holomorphic. This proves the first half of the theorem.

Let us now consider the "only if" statement. We start with two arbitrary vector fields, $X, Y \in \Gamma(TM)$ which we can decompose into (anti-)holomorphic parts

$$X = X^+ + X^-\,, \qquad Y = Y^+ + Y^-\,, \tag{3.30}$$

where

$$JX^\pm = \pm iX^\pm\,, \qquad JY^\pm = \pm iY^\pm\,. \tag{3.31}$$

By linearity (since N is a tensor field) we have

$$N(X,Y) = N(X^+,Y^+) + N(X^+,Y^-) + N(X^-,Y^+) + N(X^-,Y^-)\,, \tag{3.32}$$

and it is easy to show that

$$N(X^+, Y^-) = 0\,, \qquad N(X^-, Y^+) = 0\,. \tag{3.33}$$

As before we can show that

$$N(X^+, Y^+) = 2\left(\left[X^+, Y^+\right] + iJ\left[X^+, Y^+\right]\right)\,, \tag{3.34}$$

and by a similar calculation

$$N(X^-, Y^-) = 2\left(\left[X^-, Y^-\right] - iJ\left[X^-, Y^-\right]\right)\,. \tag{3.35}$$

However, when J is integrable both of these expressions vanish and so $N(X, Y) = 0$. This completes the proof.

Exercise 3.6: Show that

$$N(X^+, Y^-) = 0\,, \tag{3.36}$$

for any holomorphic vector field X^+ and anti-holomorphic vector field Y^-.

Because the almost complex structure on a complex manifold is constant, we find that:

Corollary: The almost complex structure on a complex manifold is integrable.

(Newlander-Nirenberg) Theorem 3.3: Let (M, J) be an almost complex manifold. If J is integrable, then (M, J) is a complex manifold.

Unfortunately, the proof is very involved and beyond the scope of these lectures. This theorem gives us an alternative way of showing that a manifold is complex.

Note: Almost complex manifolds can admit several complex structures some of which may be integrable and some not. Hence it is difficult to show that a manifold is not complex. If we can find an almost complex structure, it is not enough to show that it is not integrable. There may be another almost complex structure on the manifold that *is* integrable!

Let us use the Newlander-Nirenberg theorem to prove the following:

Theorem 3.4: Any orientable two-dimensional Riemann surface is complex.

Proof: In chapter 2, we proved that any orientable two-dimensional Riemann surface is almost complex by explicitly constructing the almost complex structure

$$J_\mu{}^\nu = \epsilon_{\mu\rho} g^{\rho\nu} \,. \tag{3.37}$$

It now remains to show that this almost complex structure is integrable. We do this by showing the Nijenhuis tensor vanishes. First notice that in two dimensions

$$\partial_{[\mu} J_{\nu]}{}^\sigma = \epsilon_{\mu\nu} V^\sigma \,, \tag{3.38}$$

where $V^\sigma = \frac{1}{2}\epsilon^{\mu\nu}\partial_\mu J_\nu{}^\sigma$. Using this result we find that

$$J_\mu{}^\sigma \partial_{[\nu} J^\rho_{\sigma]} = V^\rho g^{\sigma\lambda} \epsilon_{\lambda\mu}\epsilon_{\nu\sigma} \tag{3.39}$$

is symmetric in μ and ν. Using the expression (3.26), this immediately implies that

$$N_{\mu\nu}{}^\rho = 0 \,. \tag{3.40}$$

This completes the proof.

This is of course important for string theory, where the Euclidean worldsheet is a two-dimensional oriented Riemann surface and hence a complex manifold.

3.3 Exterior derivatives of forms

In the previous chapter we discussed the exterior derivative of (p, q)-forms on almost complex manifolds. We found that for a (p, q)-form ω

$$d\omega^{(p,q)} = (\lambda_1)^{(p+2,q-1)} + (\lambda_2)^{(p+1,q)} + (\lambda_3)^{(p,q+1)} + (\lambda_4)^{(p-1,q+2)} \,, \tag{3.41}$$

where $\lambda_1^{(p+2,q-1)}$ and $\lambda_4^{(p-1,q+2)}$ were proportional to derivatives of the projection operators $P^\pm$ (see Exercise 2.5). Thus, for a complex manifold where the complex structure J is constant, $\lambda_1^{(p+2,q-1)} = \lambda_4^{(p-1,q+2)} = 0$. Hence, for a complex manifold

$$d = \partial + \bar{\partial} \,. \tag{3.42}$$

Exercise 3.7: By using $d^2 = 0$, or otherwise, show that for a complex manifold

$$\partial^2 = \bar{\partial}^2 = 0 \,. \tag{3.43}$$

Exercise 3.8: Show that for an almost complex manifold the condition that $\partial^2 = 0$ is equivalent to the Nijenhuis tensor vanishing.

This is the starting point for Dolbeault cohomology which we will come to in a few chapter's time.