Advanced topics in string and field theory: Complex manifolds and Calabi-Yau manifolds

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4 Symplectic and Hermitian manifolds

In this chapter we will further develop the theory of complex manifolds by studying two different structures that one can define on (almost) complex manifolds. The first leads to symplectic manifolds, which underlie many areas of theoretical physics, for example classical mechanics. We will then discuss Hermitian metrics on (almost) complex manifolds and their associated fundamental forms.

4.1 Symplectic manifolds

Definition: A **symplectic manifold** (M, ω) is a manifold M equipped with a non-degenerate closed 2-form ω. Such a form is called a **symplectic form**.

The requirement that the symplectic form be non-degenerate means that $\omega_{\mu\nu}$ is invertible, i.e. it has an inverse $\omega^{\mu\nu}$ such that

$$\omega^{\mu\nu}\omega_{\nu\rho} = \delta^{\mu}{}_{\rho} \,. \tag{4.1}$$

Theorem 4.1: A symplectic manifold has even dimension.

Proof: An invertible antisymmetric matrix must have even numbers of rows and columns.

Without referring to a coordinate system, the non-degeneracy requirement is equivalent to demanding that for a manifold of dimension $2n$, the n-th wedge product of ω is nowhere vanishing:

$$\omega^n = \omega \wedge \omega \wedge \ldots \wedge \omega \neq 0 \,. \tag{4.2}$$

This condition is the same as requiring the determinant of ω to be non-zero everywhere.

Example 4.1: $\mathbb{R}^{2n}$ is symplectic. We can see this by taking $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with coordinates

x^i and y_i, where $i = 1, \dots, n$, respectively. Then the symplectic form is given by

$$\omega = dx^i \wedge dy_i \,. \tag{4.3}$$

It is clear that this is globally defined, closed and non-degenerate. In matrix notation, the symplectic form is given by

$$\omega = \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{pmatrix} \,. \tag{4.4}$$

The fact that this is reminiscent of the almost complex structure we encountered in chapter 2 (given in terms of a real coordinate basis) is not an accident. We will see the connection arise later.

Example 4.2: T^*M is symplectic. You may have encountered this example when studying classical mechanics. Let us denote by x^i local coordinates on the base manifold M, and by p_i the fibre coordinates. As a manifold T^*M has local coordinates $\left(x_i, p^i\right)$. The fibre coordinates just label the different 1-forms in T^*M, so that we can write a general 1-form as

$$\theta = p_i dx^i \,. \tag{4.5}$$

This is called the canonical 1-form and from it we can construct a symplectic form as

$$\omega = d\theta = dp_i \wedge dx^i \,. \tag{4.6}$$

By construction ω is closed and it is clearly non-degenerate. In classical mechanics, the x^i and p_i represent the position and momentum of a particle, respectively, and T^*M is the phase space of that particle.

Example 4.3: $\mathbb{CP}^N$ is symplectic. We will prove this in chapter 5 where we show that $\mathbb{CP}^N$ is Kähler and that all Kähler manifolds are symplectic.

Theorem 4.2: All 2-dimensional orientable manifolds are symplectic.

Proof: On a d-dimensional orientable manifold there exists a volume form ω which is a non-degenerate d-form. It being a top-form, the exterior derivative automatically vanishes:

$$d\omega = 0 \,. \tag{4.7}$$

In 2 dimensions, the volume form ω defines a symplectic form.

Hermitian conjugate with respect to a metric

The Hermitian conjugate $A^\dagger$ of an operator $A : TM \longrightarrow TM$ with respect to a metric g is defined as

$$g(AX, Y) = g(X, A^\dagger Y) \;\; \forall\, X, Y \in \Gamma\,(TM) \; . \tag{4.8}$$

As usual, we call A Hermitian if $A = A^\dagger$, and anti-Hermitian if $A = -A^\dagger$. In this chapter we will use two properties of Hermitian matrices. The first comes from the

Spectral theorem: If an operator is Hermitian with respect to a positive-definite inner product, then it can be diagonalised and all its eigenvalues are real.

This generalises our intuition from linear algebra. Another property we will use is that if a Hermitian matrix $A = A^\dagger$ is positive definite with respect to the positive-definite metric g, i.e.

$$g(AX, X) > 0 \;\; \forall X \neq 0 \in \Gamma\,(TM) \; , \tag{4.9}$$

then the eigenvalues of A are positive.

The most important theorem of this chapter is the that we already hinted at in example 4.1:

Theorem 4.3: Any Riemannian symplectic manifold (M, ω) is almost complex.

Proof: Let us define $A : V \longrightarrow V$ by

$$g(AX, Y) = \omega(X, Y) \;\; \forall\, X, Y \in \Gamma\,(TM) \; . \tag{4.10}$$

Using local coordiantes it is easy to see that such an A exists. We have

$$g_{\rho\nu} {A_\mu}^\rho X^\mu Y^\nu = \omega_{\mu\nu} X^\mu Y^\nu \, , \forall\, X,\, Y \in \Gamma\,(TM) \; . \tag{4.11}$$

This means that

$$g_{\rho\nu} {A_\mu}^\rho = \omega_{\mu\nu} \Rightarrow {A_\mu}^\nu = \omega_{\mu\rho} g^{\rho\nu} \, . \tag{4.12}$$

The Hermitian conjugate of A is $A^\dagger = -A$ as we can see from

$$g(AX, Y) = \omega(X, Y) = -\omega(Y, X) = -g(AY, X) = -g(X, AY) \, , \forall\, X, Y \in \Gamma\,(TM) \; . \tag{4.13}$$

This implies that $A^\dagger A = -A^2$ is a Hermitian matrix. We will now want to show that we can take an inverse square root of $A^\dagger A$ by showing that its eigenvalues are positive. To do this, we need to prove that $AA^\dagger$ is positive definite with respect to g. Note that

$$A(X) = 0 \iff X = 0 \, . \tag{4.14}$$

Thus,
$$g(A^\dagger AX, X) = g(AX, AX) > 0 \ \forall AX \neq 0 \in \Gamma(TM) \ . \tag{4.15}$$
But by using (4.14) we have that
$$g(A^\dagger AX, X) > 0 \,, \forall X \neq 0 \in \Gamma(TM) \ , \tag{4.16}$$
and hence $A^\dagger A$ is positive definite and has positive eigenvalues. Thus, we can define
$$J = \left(A^\dagger A\right)^{-1/2} A \,, \tag{4.17}$$
which because $A^\dagger = -A$ satisfies
$$J^2 = \left(A^\dagger A\right)^{-1} A^2 = -\mathbb{I} \,. \tag{4.18}$$
J is globally well-defined because ω and g are and thus is an almost complex structure. This completes our proof.

Exercise 4.1: Starting from (4.8), show that in a local coordinate basis the Hermitian conjugate is defined as
$$\left(A^\dagger\right)_\mu{}^\nu = g_{\mu\sigma} g^{\nu\rho} A_\rho{}^\sigma \ . \tag{4.19}$$
We see that $A^\dagger$ always exists.

Exercise 4.2: Show that $A(X) = 0 \iff X = 0$.
Hint: Let $AX = 0$ for some X and use the definition of A and non-degeneracy of ω to show that this implies $X = 0$.

An acute reader may have spotted that the almost complex structure defined here (4.17) is the generalisation of the almost complex structure defined in chapter 2 for any orientable 2-dimensional Riemann surface (which by Theorem 4.2 is a symplectic manifold).

Exercise 4.3: In chapter 2 we constructed an almost complex structure for any orientable 2-dimensional Riemann surface. Show that in this case the A defined above is

$$A_\mu{}^\nu = J_\mu{}^\nu = \epsilon_{\mu\rho} g^{\rho\nu} \,. \tag{4.20}$$

Furthermore, show explicitly using the local coordinate expression (4.19) that in this case A is anti-Hermitian, i.e. that

$$\left(A^\dagger\right)_\mu{}^\nu \equiv g_{\mu\sigma} g^{\nu\rho} A_\rho{}^\sigma = -A_\mu{}^\nu \,. \tag{4.21}$$

Finally show that in this example

$$\left(A^\dagger A\right)_\mu{}^\nu = \delta_\mu{}^\nu \,. \tag{4.22}$$

Definition: An almost complex structure J on a symplectic manifold (M, ω) is said to be **compatible** if for all $X, Y \in \Gamma(TM)$

$$\omega(JX, JY) = \omega(X, Y) \,, \qquad \omega(X, JX) > 0 \text{ if } X \neq 0 \,. \tag{4.23}$$

Note: The compatibility requirement is defined with respect to *real* vectors.

Corollary: The almost complex structure J in (4.17) is compatible with ω.
Proof: First of all, notice that $J^\dagger = -J$ is anti-Hermitian and that $JA = AJ$. Then, we can show that for any $X, Y \in \Gamma(TM)$

$$\omega(JX, JY) = g(AJX, JY) = g(JAX, JY) = -g\left(AX, J^2 Y\right) = g(AX, Y) = \omega(X, Y) \,. \tag{4.24}$$

We can also use the positive-definiteness of $A^\dagger A$ to show that for any $X \neq 0 \in \Gamma(TM)$

$$\omega(X, JX) = g(AX, JX) = -g(JAX, X) = g\left(\sqrt{A^\dagger A} X, X\right) > 0 \,. \tag{4.25}$$

This completes our proof.

4.2 Hermitian manifolds

Definition: Let (M, J) be a $2n$-dimensional (almost) complex manifold, and let g be a Riemannian metric on M. The metric g is an **(almost) Hermitian metric** if it satisfies any of the three following equivalent conditions:

- $g(X, Y) = g(JX, JY) \ \forall X, Y \in \Gamma(TM)$, (4.26)
- $g_{\mu\nu} = J_\mu{}^\rho J_\nu{}^\sigma g_{\rho\sigma}$, (4.27)
- In local complex coordinates z^a, $\bar{z}^a$, where $a, b = 1, \ldots, n$,

$$g = g_{a\bar{b}}(z, \bar{z}) \left(dz^a \otimes d\bar{z}^b + d\bar{z}^b \otimes dz^a \right) , \tag{4.28}$$

  i.e. the components $g_{ab} = g_{\bar{a}\bar{b}} = 0$ vanish.

Exercise 4.4: Show that the first two conditions, (4.26) and (4.27), are equivalent.

It is easy to show that the first condition implies that any (anti-)holomorphic vector fields are orthogonal with respect to the metric, i.e. if $X, Y \in \Gamma(TM^+)$ so that

$$JX = iX , \qquad JY = iY , \tag{4.29}$$

then

$$g(X, Y) = 0 , \tag{4.30}$$

and similarly for anti-holomorphic vector fields.

Exercise 4.5: Prove the above, i.e. that any two holomorphic vector fields are orthogonal with respect to g.

Since $\frac{\partial}{\partial z^a}$ and $\frac{\partial}{\partial \bar{z}^a}$ are a basis for holomorphic and anti-holorphic vectors, respectively, we find that the components

$$g_{ab} = 0 , \qquad g_{\bar{a}\bar{b}} = 0 , \tag{4.31}$$

and hence that the first condition implies the third.

Let us now show the converse, i.e. that if any (anti-)holomorphic vector fields are orthogonal with respect to the metric then this implies $g(X, Y) = g(JX, JY)$ for any $X, Y \in \Gamma(TM)$.
Proof: Recall that we can write any $X, Y \in \Gamma(TM)$ as

$$X = X^+ + X^- , \qquad Y = Y^+ + Y^- , \tag{4.32}$$

where $JX^{\pm} = \pm X^{\pm}$ and similarly for Y. Then,

$$g(X,Y) = g\left(X^{+},Y^{-}\right) + g\left(X^{-},Y^{+}\right) , \tag{4.33}$$

Now instead of considering the vector fields X and Y, consider the vector fields JX and JY. We know that

$$P^{\pm}JX = \pm i P^{\pm}X\,, \tag{4.34}$$

and thus

$$g(JX,JY) = i\,(-i)\,g\left(X^{+},Y^{-}\right) + (-i)\,i\,g\left(X^{-},Y^{+}\right) = g\left(X^{+},Y^{-}\right) + g\left(X^{-},Y^{+}\right) = g(X,Y)\ . \tag{4.35}$$

This completes the proof.

Exercise 4.6: Repeat the proof above in local coordinates.

We see that a Hermitian metric defines a positive-definite inner product between the holomorphic and anti-holomorphic tangent spaces:

$$g : T_pM^{+} \otimes T_pM^{-} \longrightarrow \mathbb{C}\ \forall\, p \in M\,. \tag{4.36}$$

Note: Hermiticity is a restriction on the metric, **not** on the manifold. In particular:

Theorem 4.4: Any (almost) complex manifold (M,J) admits an (almost) Hermitian metric.

Proof: Let g be a Riemannian metric on M and define for any $X,Y \in \Gamma(TM)$,

$$h(X,Y) = \frac{1}{2}\Big(g(X,Y) + g(JX,JY)\Big)\,. \tag{4.37}$$

Exercise 4.7: Show that for any $X,Y \in \Gamma(TM)$,

$$h(JX,JY) = h(X,Y)\,, \tag{4.38}$$

and that

$$h(X,X) \geq 0\,, \tag{4.39}$$

with equality if and only if $X = 0$.

This completes the proof.

Let us stress that in the above we consider only *real* vectors. On the complexified tangent space, the metric is not positive-definite.

Corollary: A symplectic manifold always admits an almost Hermitian metric.
Proof: This follows immediately from theorem 4.3 which states that every symplectic manifold is almost complex.

Definition: Let (M, J) be an (almost) complex manifold with a Hermitian metric g. Then we define the **fundamental 2-form** as

$$\omega(X, Y) = g(JX, Y) \ \forall X, Y \in \Gamma(TM) \ . \tag{4.40}$$

This is also sometimes called the **Hermitian 2-form**.

Let us first show that this is actually a 2-form. For any $X, Y \in \Gamma(TM)$ we find

$$\omega(X, Y) = g(JX, Y) = g\left(J^2 X, JY\right) = -g(X, JY) = -g(JY, X) = -\omega(Y, X) \ . \tag{4.41}$$

In a coordinate basis the definition implies

$$\omega_{\mu\nu} = J_\mu{}^\rho g_{\rho\nu} \ , \tag{4.42}$$

and hence in complex coordinates we find

$$\omega = i g_{a\bar{b}} \left(dz^a \otimes d\bar{z}^b - d\bar{z}^b \otimes dz^a\right) \ , \tag{4.43}$$

and we see that it is a $(1,1)$-form.

Note: You will sometimes see the convention amongst physicists to label the components of $\omega_{\mu\nu}$ as $J_{\mu\nu} \equiv \omega_{\mu\nu} = J_\mu{}^\rho g_{\rho\nu}$.

To get more familiar with this 2-form let us study a couple of properties of ω.
Corollary: The fundamental 2-form is non-degenerate.
Proof: Use local coordinates and define

$$\omega^{\mu\nu} = -g^{\mu\rho} J_\rho{}^\nu = J_\rho{}^\mu g^{\rho\nu} \ . \tag{4.44}$$

Then this defines an inverse since

$$\omega^{\mu\nu}\omega_{\nu\rho} = -g^{\mu\sigma} J_\sigma{}^\nu J_\nu{}^\kappa g_{\kappa\rho} = \delta^\mu{}_\rho \ . \tag{4.45}$$

This completes the proof.

Exercise 4.8: Show that $g^{\mu\rho} J_\rho{}^\nu = -J_\rho{}^\mu g^{\rho\nu}$.

Corollary: The fundamental form ω is compatible with J.
Proof: Recall that compatibility requires for any $X, Y \in \Gamma(TM)$ that $\omega(JX, JY) = \omega(X, Y)$ and $\omega(X, JX) > 0$ when $X \neq 0$. Let us prove the first requirement:

$$\omega(JX, JY) = g\left(J^2 X, JY\right) = -g(X, JY) = -g\left(JX, J^2 Y\right) = g(JX, Y) = \omega(X, Y) \ . \quad (4.46)$$

The second requirement follows from

$$\omega(X, JX) = g(JX, JX) = g(X, X) > 0 \,, \text{ if } X \neq 0 \,. \quad (4.47)$$

This completes our proof.