Advanced topics in string and field theory: Complex manifolds and Calabi-Yau manifolds

Emanuel Malek

5 Kähler manifolds

In this chapter we will further develop the concepts met in the previous chapter which will lead us to Kähler manifolds. This special kind of complex manifolds plays an important role in string theory. In order to get ready for Calabi-Yau manifolds, we will also discuss the holonomy of Kähler manifolds.

Let us start by defining a Kähler manifold.

Definition: Let (M, J) be a complex manifold with Hermitian matrix g and fundamental 2-form ω. If ω is closed, i.e.

$$d\omega = 0\,, \tag{5.1}$$

then M is called a **Kähler manifold**, g is called the **Kähler metric** and ω the **Kähler form**.

Note: When (M, g, J) is an almost Hermitian manifold with $d\omega = 0$ then M is called **almost Kähler**.

Example 5.1: All complex manifolds of 2 (real) dimensions are Kähler. This is because all complex manifolds are Hermitian and any 2-form ω on a 2 dimensional manifold is closed.

Corollary: An (almost) Kähler manifold is a symplectic manifold.

How about the inverse? When is an (almost) complex, symplectic manifold Kähler? The answer lies in the compatibility between ω and J.

Theorem 5.1: Let (M, ω, J) be a symplectic manifold with a compatible, (almost) complex structure J. Then M is Kähler.

Proof: Construct the metric

$$g\,(X, Y) = -\omega\,(JX, Y)\;. \tag{5.2}$$

Exercise 5.1: Show that the metric defined above (5.2) satisfies

(a) $$g\,(X,Y) = g\,(Y,X),$$

(b) $$g\,(JX,JY) = g\,(X,Y),$$

(c) $$g\,(X,X) > 0 \text{ for all } X \neq 0.$$

Hint: ω is compatible with J.

Thus, g as defined above is a positive-definite Hermitian metric, and since the symplectic two-form ω is closed, the manifold is (almost) Kähler. This completes the proof.

Theorem 5.2: Any closed (p,q)-form ω can be written locally as

$$\omega = \partial\bar{\partial}\chi\,, \tag{5.3}$$

for some $(p-1,q-1)$-form χ.

Proof: This is just a re-statement of the Poincaré Lemma in complex coordinates. Firstly, note that

$$d\omega = \partial\omega + \bar{\partial}\omega = 0\,, \tag{5.4}$$

implies that the $(p+1,q)$ and $(p,q+1)$ forms

$$\partial\omega = \bar{\partial}\omega = 0\,, \tag{5.5}$$

vanish separately. We can write the (p,q) form ω in local coordinates as

$$\omega = \frac{1}{(p+q)\,!}\omega_{a_1\ldots a_p\bar{b}_1\ldots\bar{b}_q}dz^{a_1}\wedge\ldots dz^{a_p}\wedge d\bar{z}^{b_1}\wedge\ldots\wedge d\bar{z}^{b_q}\,, \tag{5.6}$$

and so

$$\partial\omega = \frac{1}{(p+q)\,!}\partial_{c_1}\omega_{a_1\ldots a_p\bar{b}_1\ldots\bar{b}_q}dz^{c_1}\wedge dz^{a_1}\wedge\ldots\wedge dz^{a_p}\wedge d\bar{z}^{b_1}\wedge\ldots\wedge d\bar{z}^{b_q}\,, \tag{5.7}$$

and similarly for $\bar{\partial}\omega$. We thus have

$$\partial_{[c_1}\omega_{a_1\ldots a_p]\bar{b}_1\ldots\bar{b}_q} = 0\,, \qquad \partial_{[\bar{c}_1}\omega_{|a_1\ldots a_p|\bar{b}_1\ldots\bar{b}_q]} = 0\,. \tag{5.8}$$

These then imply by the Poincaré Lemma that locally

$$\omega = \partial\bar{\partial}\chi\,, \tag{5.9}$$

for some $(p-1,q-1)$-form $\chi(z,\bar{z})$.

Although what follows is a simple consequence of the above theorem, it is important enough for string theorists that I will label it as a "theorem".

Theorem 5.3: For a Kähler manifold, the Kähler metric can locally be written as

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K \,, \tag{5.10}$$

where $K(z, \bar{z})$ is some scalar function, known as the **Kähler potential**.

Proof: For a Kähler manifold, the Kähler form is closed and hence can be locally be written as

$$\omega_{a\bar{b}} = i\partial_a \partial_{\bar{b}} K \,, \tag{5.11}$$

for some scalar $K(z, \bar{z})$.[1] Recall from chapter 4 that in local coordinates, the fundamental form is given in terms of the metric as

$$\omega = ig_{a\bar{b}} \left(dz^a \otimes d\bar{z}^b - d\bar{z}^b \otimes dz^a \right) \,, \tag{5.12}$$

and hence we see that Kähler metric can expressed in terms of the Kähler potential as

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K \,. \tag{5.13}$$

This completes our proof.

The Kähler potential is defined locally. Let K_i be the Kähler potential in a patch U_i. On a non-trivial overlap of two patches $U_i \cap U_j \neq \emptyset$, the Kähler potentials K_i and K_j must be related by a **Kähler transformation**

$$K_i \left(z, \bar{z} \right) = K_j \left(z, \bar{z} \right) + f_{ij} \left(z \right) + \bar{f}_{ij} \left(\bar{z} \right) \,, \tag{5.14}$$

where $f_{ij} \left(z \right)$ is a holomorphic function. This clearly does not change the metric.

Note: In practice we can show that a complex manifold is Kähler by finding a globally defined Kähler potential (up Kähler transformations) on that complex manifold and then showing that the resultant Kähler metric is positive definite. Let us use this technique in the following example.

Example 5.2: $\mathbb{CP}^N$ is a Kähler manifold. We showed in Chapter 3 that $\mathbb{CP}^N$ is complex by using an atlas $\left(U_i, \xi^a_{[i]} \right)$ with coordinates

$$\xi^a_{[i]} = \frac{z^a}{z^i} \,, \tag{5.15}$$

where z^a are (homogeneous) coordinates of $\mathbb{C}^{N+1}$ defined in the patches U_i consisting of all

[1] The factor of i is just to make the following equations neater.

points where $z^i \neq 0$. Now, consider the function

$$K_i = \ln \left(\sum_{a=1}^{N+1} |\xi^a_{[i]}|^2 \right) , \tag{5.16}$$

defined on a patch U_i. On the intersection of two patches $U_i \cap U_j \neq \emptyset$, the coordinates are related by

$$\xi^a_{[i]} = \frac{\xi^a_{[j]}}{\xi^i_{[j]}} , \tag{5.17}$$

and hence the functions K_i and K_j are related by

$$K_i \left(z, \bar{z} \right) = K_j \left(z, \bar{z} \right) - \ln \xi^i_{[j]} - \ln \bar{\xi}^i_{[j]} . \tag{5.18}$$

This is a Kähler transformation and hence we can globally define a metric

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K_i = \partial_a \partial_{\bar{b}} K_j , \tag{5.19}$$

and similarly for the Kähler form. Here we abused notation and labelled both the complex coordinates on $\mathbb{C}^{N+1}$, z^a, and the inhomogeneous complex coordinates on $\mathbb{CP}^N$, ξ^a, by the same label a. For the $\mathbb{CP}^N$ case, we really mean $a = 1, \ldots, N$ since it is an N-dimensional complex manifold and clearly $\xi^{N+1} = 1$ as defined above. This is common abuse of notation.

We need to now show that the metric, called the **Fubini-Study** metric, is positive definite on real vectors. Let us evaluate it in a coordinate patch $\left(U_i, \xi^a_{[i]} \right)$ and let us drop the label i for the patch. We find

$$g_{a\bar{b}} = \frac{\delta_{a\bar{b}} \left(1 + |\xi|^2 \right) - \bar{\xi}_a \xi_b}{\left(1 + |\xi|^2 \right)^2} . \tag{5.20}$$

Now let us show that this is positive-definite on TM. Let $X \in \Gamma \left(TM \right)$ be a real vector-field, then we know that

$$X^{\bar{a}} = \overline{X^a} , \tag{5.21}$$

and hence

$$g_{\mu\nu} X^\mu X^\nu = \frac{|X|^2 + |X|^2 |\xi|^2 - |\left(\bar{\xi} X \right)|^2}{\left(1 + |\xi|^2 \right)^2} . \tag{5.22}$$

By the Schwarz Inequality we know that

$$|X|^2 |\xi|^2 - |\left(\bar{\xi} X \right)|^2 > 0 , \tag{5.23}$$

and thus we find that g is positive definite. This completes the proof that $\mathbb{CP}^N$ is Kähler.

Exercise 5.2: Derive the Fubini-Study metric (5.20) from the Kähler potential (5.16) in the patch U_{N+1} where it is given by

$$K = \ln\left(1 + \xi^1\bar{\xi}^1 + \ldots + \xi^N\bar{\xi}^N\right) . \tag{5.24}$$

5.1 Back to Holonomy

We saw that Hermitian and Kähler metrics have a restricted shape: only their "mixed" components $g_{a\bar{b}}$ are non-zero. This strongly constrains the Riemann curvature tensor and hence the holonomy group as we will see in this subsection.

5.1.1 Connection and curvature of Hermitian manifolds

Let us begin by discussing Hermitian metrics before further imposing that the metric be Kähler (i.e. that the fundamental form be closed). To study the curvature of a manifold we have to specify the connection to be used. Usually, we take this to be the Levi-Civita connection which is the unique metric-compatible torsion-free connection. However, here we will first use a *different* connection known as the Chern connection. Let us define it and show it is unique.

Definition: Let (M, J, g) a Hermitian manifold. Then we can construct a connection that is compatible with the Hermitian metric and the complex structure, i.e.

$$\nabla g = \nabla J = 0 . \tag{5.25}$$

This is called a **Hermitian connection**.

Note: Hermitian connections are not unique!

We need to impose another constraint to obtain a unique connection, known as the Chern connection.

Theorem 5.4: On a Hermitian manifold there exists a *unique* Hermitian connection called the **Chern connection** with the added property that the anti-holomorphic covariant derivative of a holomorphic vector field is just given by the anti-holomorphic derivative of the holomorphic vector field, and similarly for the holomorphic covariant derivative of an anti-holomorphic vector field, i.e. in local complex coordinates

$$\nabla_{\bar{a}} V^b = \partial_{\bar{a}} V^b , \qquad \nabla_a V^{\bar{b}} = \partial_a V^{\bar{b}} . \tag{5.26}$$

Proof: Because $\nabla J = 0$ we find that

$$\Gamma_{ab}{}^{\bar{c}} = 0 , \qquad \Gamma_{\bar{a}b}{}^{\bar{c}} = 0 , \qquad \Gamma_{a\bar{b}}{}^{c} = 0 , \qquad \Gamma_{\bar{a}\bar{b}}{}^{c} = 0 . \tag{5.27}$$

The extra requirement for the Chern connection then implies that

$$\Gamma_{\bar{a}b}{}^{c} = 0\,, \qquad \Gamma_{a\bar{b}}{}^{\bar{c}} = 0\,, \tag{5.28}$$

and hence the only non-vanishing components of the connection have "pure indices" (meaning all of one kind):

$$\Gamma_{ab}{}^{c} \neq 0\,, \qquad \Gamma_{\bar{a}\bar{b}}{}^{\bar{c}} \neq 0\,. \tag{5.29}$$

Let us now impose that the metric must be covariantly constant. Thus,

$$\nabla_a g_{b\bar{c}} = \partial_a g_{b\bar{c}} - \Gamma_{ab}{}^{d} g_{d\bar{c}} = 0\,, \qquad \nabla_{\bar{a}} g_{b\bar{c}} = \partial_{\bar{a}} g_{b\bar{c}} - \Gamma_{\bar{a}\bar{c}}{}^{\bar{d}} g_{b\bar{d}} = 0\,. \tag{5.30}$$

We can invert these expressions to find

$$\Gamma_{ab}{}^{c} = g^{c\bar{d}} \partial_a g_{\bar{d}b}\,, \qquad \Gamma_{\bar{a}\bar{b}}{}^{\bar{c}} = g^{\bar{c}d} \partial_{\bar{a}} g_{d\bar{b}}\,. \tag{5.31}$$

This completes our proof by construction.

Exercise 5.3: Show that $\nabla J = 0$ implies (5.27).

Exercise 5.4: Show that for an arbitrary connection $\Gamma_{\mu\nu}{}^{\rho}$, the "mixed" components of the connection are tensors under a holomorphic change of coordinates.
Hint: Recall that under an infinitesimal diffeomorphism $x^\rho \to x^\rho - \xi^\rho$, a connection transforms as $\delta_\xi \Gamma_{\mu\nu}{}^{\rho} = \partial_\mu \partial_\nu \xi^\rho$.

We can now show that the only non-zero components of the Riemann tensor are

$$R^{a}{}_{bc\bar{d}} = -\partial_{\bar{d}} \Gamma_{cb}{}^{a}\,, \qquad R^{\bar{a}}{}_{\bar{b}\bar{c}d} = -\partial_{d} \Gamma_{\bar{c}\bar{b}}{}^{a}\,. \tag{5.32}$$

Exercise 5.5: Using the definition of the Riemann tensor in a local coordinate basis as

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma_{\sigma\nu}{}^{\mu} + \Gamma_{\rho\lambda}{}^{\mu} \Gamma_{\sigma\nu}{}^{\lambda} - (\rho \leftrightarrow \sigma)\,, \tag{5.33}$$

and the fact that the connection is non-zero only when it has pure indices, show that

$$R^{\bar{a}}{}_{b\mu\nu} = 0\,, \qquad R^{a}{}_{\bar{b}\mu\nu} = 0\,, \qquad R^{a}{}_{b\bar{c}\bar{d}} = 0\,, \qquad R^{\bar{a}}{}_{\bar{b}cd} = 0\,. \tag{5.34}$$

Also show using (5.31) that

$$R^{a}{}_{bcd} = 0\,, \qquad R^{\bar{a}}{}_{\bar{b}\bar{c}\bar{d}} = 0\,. \tag{5.35}$$

In ordinary differential geometry, the symmetry properties of the Riemann curvature tensor imply that we can only form one tensor (the Ricci tensor) by contracting its indices. However, a complex structure gives us an alternative way of contracting indices and this allows us to define the Ricci 2-form.

Definition: Let (M, J, g) be a Hermitian manifold and let us label the components of the Riemann curvature tensor in local coordinates as $R^{\mu}{}_{\nu\rho\sigma}$. We define the **Ricci form** as

$$\mathcal{R} = \frac{1}{4} R^{\mu}{}_{\nu\rho\sigma} J^{\nu}{}_{\mu} dx^{\rho} \wedge dx^{\sigma} \,. \tag{5.36}$$

Theorem 5.5: Let (M, J, g) be a Hermitian manifold. Then its Ricci form is given by

$$\mathcal{R} = -i\partial\bar{\partial} \ln \sqrt{|g|} \,. \tag{5.37}$$

Proof: We will use local coordinates throughout to evaluate the Ricci form. Observe from (5.32) that only the mixed components of $\mathcal{R}_{a\bar{b}}$ are non-zero and given by

$$\mathcal{R}_{a\bar{b}} = \frac{i}{2} \left(R^{c}{}_{ca\bar{b}} - R^{\bar{c}}{}_{\bar{c}a\bar{b}} \right) \,. \tag{5.38}$$

In order to evaluate them, note that

$$\partial_{\mu} \ln|g| = g^{\nu\rho} \partial_{\mu} g_{\nu\rho} = 2 g^{a\bar{b}} \partial_{\mu} g_{a\bar{b}} \,, \tag{5.39}$$

Let us thus evaluate

$$R^{c}{}_{ca\bar{b}} = -\partial_{\bar{b}} \Gamma_{ac}{}^{c} = -\partial_{\bar{b}} \left(g^{c\bar{d}} \partial_a g_{\bar{d}c} \right) = -\partial_{\bar{b}} \partial_a \ln \sqrt{|g|} \,, \tag{5.40}$$

and similarly

$$R^{\bar{c}}{}_{\bar{c}a\bar{b}} = \partial_a \partial_{\bar{b}} \ln \sqrt{|g|} \,. \tag{5.41}$$

Using these results we find that

$$\mathcal{R}_{a\bar{b}} = -i \partial_a \partial_{\bar{b}} \ln \sqrt{|g|} \,. \tag{5.42}$$

This completes the proof.

Exercise 5.6: $\sqrt{|g|}$ is a scalar density and transforms as

$$\sqrt{|g|} \longrightarrow \sqrt{|g|}|Jac|\,, \tag{5.43}$$

under a coordinate transformation where Jac is the Jacobian of the coordinate transformations. Show that

$$\mathcal{R} = -i\partial\bar{\partial}\ln\sqrt{|g|} \tag{5.44}$$

is nonetheless globally well-defined, i.e. it transforms as a tensor.
Hint: Recall that coordinate transformations are holomorphic.

Let us take a brief excursion to cohomology which we will study in more detail in the next chapter.

Exercise 5.7: Show that on a complex manifold,

$$\partial\bar{\partial} = -\frac{1}{2}d\left(\partial - \bar{\partial}\right)\,. \tag{5.45}$$

Hint: Use the fact that on a complex manifold $d = \partial + \bar{\partial}$.

Using the result (5.45) we see that

$$\mathcal{R} = -i\partial\bar{\partial}\ln\sqrt{|g|} = \frac{i}{2}d\left(\partial - \bar{\partial}\right)\ln\sqrt{|g|}\,. \tag{5.46}$$

Recall that we call a p-form $\omega = d\kappa$ for some $(p-1)$-form κ is exact. This seems to suggest that $\mathcal{R}$ is exact. However, this is not the case since $\sqrt{|g|}$ is not a coordinate scalar, as explained in exercise 5.6. Thus, $\mathcal{R}$ is not exact. It is still closed, hence

$$d\mathcal{R} = 0\,. \tag{5.47}$$

As will be made clearer in the next chapter, we can define a cohomology class, called the Chern class from the Ricci-form.

Definition: The **first Chern class** c_1 is the space of two-forms $\frac{1}{2\pi}\mathcal{R}'$ such that

$$\frac{1}{2\pi}\mathcal{R}' - \frac{1}{2\pi}\mathcal{R} = d\kappa\,, \tag{5.48}$$

for some one-form κ, i.e. the difference between $\mathcal{R}'$ and the Ricci-form is exact. This is often written as:

$$c_1 = \left[\frac{1}{2\pi}\mathcal{R}\right]\,. \tag{5.49}$$

We will see soon that the first Chern class is the only topological obstruction to having a Calabi-Yau manifold, i.e. that a Kähler manifold with vanishing first Chern class is Calabi-Yau.

Exercise 5.8: Show that the Ricci form of the Fubini-Study metric of $\mathbb{CP}^N$ is given by

$$\mathcal{R} = -(N+1)\,\omega\,, \tag{5.50}$$

where ω is the Kähler form of $\mathbb{CP}^N$.
Hint: Use the fact that the inverse of the Fubini-Study metric is given by

$$g^{a\bar{b}} = \left(1+|\xi|^2\right)\left(\delta^{a\bar{b}} + \xi^a\bar{\xi}^b\right)\,, \tag{5.51}$$

and compute

$$\partial \ln \sqrt{|g|} = g^{a\bar{b}}\partial g_{\bar{b}a}\,. \tag{5.52}$$

5.1.2 Curvature and connection of Kähler manifolds

Let us now study the connection and curvature on a Kähler manifold.

Theorem 5.6: On a Kähler manifold, the Chern connection is the Levi-Civita connection.

Proof: Recall that a Kähler manifold has a closed fundamental form ω, i.e.

$$d\omega = 0\,. \tag{5.53}$$

From (5.12) we see that in local coordinates this implies

$$\partial_a g_{b\bar{c}} - \partial_b g_{a\bar{c}} = 0\,, \qquad \partial_{\bar{a}} g_{\bar{b}c} - \partial_{\bar{b}} g_{\bar{a}c} = 0\,. \tag{5.54}$$

Using this result and the expression for the Chern connection (5.31) we see that

$$\Gamma_{ab}{}^{c} = g^{c\bar{d}}\partial_a g_{\bar{d}b} = g^{c\bar{d}}\partial_b g_{\bar{d}a} = \Gamma_{ba}{}^{c}\,, \tag{5.55}$$

and similarly for $\Gamma_{\bar{a}\bar{b}}{}^{\bar{c}}$. Hence, the Chern connection has vanishing torsion $\Gamma_{[ab]}{}^{c}$. However, we know that there is a unique metric-compatible, torsion-free connection on any manifold: the Levi-Civita connection. This completes our proof.

Corollary: Let (M, J) be a Kähler manifold. Then the complex structure is covariantly constant (with respect to the Levi-Civita connection).

Theorem 5.7: Let (M, J, g) be a Kähler manifold. In local coordinates its Ricci tensor and Ricci form are related by

$$R_{a\bar{b}} = -i\mathcal{R}_{a\bar{b}}\,, \tag{5.56}$$

and all the other components of the Ricci tensor vanish.

Proof: Recall that in local coordinates the Ricci tensor is given by

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}\,. \tag{5.57}$$

. From (5.34) and (5.35) we immediately see that the pure components vanish:

$$R_{ab} = R_{\bar{a}\bar{b}} = 0\,. \tag{5.58}$$

Thus, the only non-zero components are then the mixed ones given by

$$R_{a\bar{b}} = R^{c}{}_{ac\bar{b}} = -\partial_{\bar{b}}\Gamma_{ca}{}^{c} = -\partial_{\bar{b}}\Gamma_{ac}{}^{c}\,, \tag{5.59}$$

where in the last equality we used the fact that the torsion vanishes. Comparing to the expression for the Ricci form we find that

$$R_{a\bar{b}} = -i\mathcal{R}_{a\bar{b}}\,. \tag{5.60}$$

This completes the proof.

We can now see why we called $\mathcal{R}$ the "Ricci-form": on a Kähler manifold, it contains the same information as the Ricci tensor. You may find it strange that this could be true of a symmetric and antisymmetric tensor. This works here because the pure components of each tensor vanish so on a $(d = 2n)$-dimensional manifold they both only contain n^2 independent components.

5.1.3 Connection to holonomy

Now that we know the Riemann curvature tensor, we can study the holonomy group of Kähler manifolds. Recall the definition of the Riemann curvature tensor.

Definition: For any three vector fields $X, Y, Z \in \Gamma(TM)$, the Riemann curvature tensor is defined as

$$R(X, Y)\,Z = [\nabla_X, \nabla_Y]\,Z - \nabla_{[X,Y]}Z\,. \tag{5.61}$$

In local coordinates this implies for any $V \in \Gamma(TM)$

$$[\nabla_\mu, \nabla_\nu]\,V^\rho = R^{\rho}{}_{\sigma\mu\nu}V^\sigma\,. \tag{5.62}$$

Thus, when parallel transporting the vector around an infinitesimally small loop (which is the same as taking the commutator of covariant derivatives) any vector gets mapped to

$$V'^{\mu} = V^{\mu} + a^{\rho\sigma} R_{\nu\rho\sigma}{}^{\mu} V^{\nu} \,, \tag{5.63}$$

where $a^{\rho\sigma}$ depends on the loop chosen.

Theorem 5.8: The holonomy group of a $2n$-dimensional Kähler manifold is $U(n)$.

Proof: From (5.63) above we can identify the elements of the Lie algebra of the holonomy group with the components of the Riemann curvature tensor as

$$\left(M_{\mu\nu}\right)_{\rho}{}^{\sigma} \equiv R^{\sigma}{}_{\rho\mu\nu} \,, \tag{5.64}$$

where $M_{\mu\nu}$ are the individual Lie algebra generators. These are in general elements of the Lie algebra of $SO(2n)$. It is an exercise in group theory (see exercise 5.9 below) to show that if

$$M_{a}{}^{\bar{b}} = 0 \,, \qquad M_{\bar{a}}{}^{b} = 0 \,, \tag{5.65}$$

then they are elements of $U(n)$. From (5.34),(5.35) we see that this is the case here and hence the holonomy group of a $2n$-dimensional Kähler manifold is $U(n)$. This completes the proof.

Exercise 5.9: Holomorphic transition functions of $2n$ complex coordinates form the group $U(n) \subset SO(2n)$. By considering the action of the matrix

$$M = \begin{pmatrix} M_{a}{}^{b} & M_{a}{}^{\bar{b}} \\ M_{\bar{a}}{}^{b} & M_{\bar{a}}{}^{\bar{b}} \end{pmatrix} \tag{5.66}$$

on the complex coordinates

$$\begin{pmatrix} z^{a} \\ \bar{z}^{a} \end{pmatrix} , \tag{5.67}$$

show that holomorphicity requires $M_{a}{}^{\bar{b}} = M_{\bar{a}}{}^{b} = 0$. We can see that the $U(n)$ is embedded in $SO(2n)$ via the matrices $M_{a}{}^{b}$ and $M_{\bar{a}}{}^{\bar{b}}$. Further requiring the matrix M to be real implies that

$$M_{\bar{a}}{}^{\bar{b}} = \overline{M_{a}{}^{b}} \,. \tag{5.68}$$

Of course, we can write $U(n) = SU(n) \times U(1)$. In chapter 1 we saw that we are particularly interested in $SU(n)$ holonomy. Can we easily see what the $U(1)$ corresponds to?

Theorem 5.9: An n-dimensional Kähler manifold with $SU(n)$ holonomy has vanishing Ricci tensor.

Proof: Above we saw that $U(n)$ is embedded in $SO(2n)$ via the matrices ${M_a}^b$ and ${M_{\bar{a}}}^{\bar{b}}$ so that an element $M \in SO(2n)$ can be written as

$$ {M_\mu}^\nu = \begin{pmatrix} {M_a}^b & 0 \\ 0 & {M_{\bar{a}}}^{\bar{b}} \end{pmatrix} . \tag{5.69} $$

We can now decompose the $U(n)$ generator ${M_a}^b$ and its complex conjugate ${M_{\bar{a}}}^{\bar{b}}$ in terms of a (traceless) $SU(n)$ generator ${H_a}^b$ and a $U(1)$ generator θ as follows.

$$ {M_a}^b = {H_a}^b + i{\delta_a}^b\theta/n \,, \qquad {M_{\bar{a}}}^{\bar{b}} = {H_{\bar{a}}}^{\bar{b}} - i{\delta_{\bar{a}}}^{\bar{b}}\theta/n \,. \tag{5.70} $$

This may be more reminiscent when written in terms of group elements:

$$ {M_a}^b = {H_a}^b e^{i\theta/n} \,, \qquad |I|= 1 \,. \tag{5.71} $$

We thus see that the $U(1)$ part corresponds to the trace of ${M_a}^b$, i.e.

$$ i\theta = {M_a}^a \,. \tag{5.72} $$

Translating this back to the $SO(2n)$ elements we find

$$ \theta = -\frac{1}{2}{J_\mu}^\nu {M_\nu}^\mu \,, \tag{5.73} $$

and hence we can identify the $U(1)$ generator with

$$ \theta = -\frac{1}{2}R^\sigma{}_{\rho\mu\nu} {J_\sigma}^\rho = -\mathcal{R}_{\mu\nu} \,. \tag{5.74} $$

We see that the Ricci-form and hence the Ricci-tensor generate the $U(1)$ part of the holonomy group of a Kähler manifold. Thus vanishing Ricci-tensor implies $SU(n)$ holonomy. This completes the proof.