Advanced topics in string and field theory: Complex manifolds and Calabi-Yau manifolds

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6 Cohomology and homology

We are now in a position to state what a Calabi-Yau manifold is but we will delay this until the final chapter in order to first introduce the tools of cohomology and homology. We will see that these are extremely useful when studying Calabi-Yau manifolds and also in order to understand the low-energy theories arising from compactifications on any internal manifold (or orbifold), as we will make clear in the final chapter of this lecture series. Finally, cohomology has many applications in theoretical physics, such as gauge theories or supersymmetry and so is useful in its own right.

6.1 Cohomology

Let us begin by discussing cohomology. I will assume some familiarity with differential forms, in particular integrating forms over manifolds but will introduce many of the concepts that we will need.

Firstly, let us label by $\Omega^p(M)$ the space of smooth p-forms. Let us define objects which should be familiar from differential geometry.

Definition: Let M be a differentiable manifold. Then we define the **exterior derivative** acting on p-forms as

$$d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M) \,. \tag{6.1}$$

For a p-form ω, with components in local coordinates

$$\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \,, \tag{6.2}$$

we define the action of d by

$$d\omega = \frac{1}{p!}\partial_{[\mu_1}\omega_{\mu_2\ldots\mu_{p+1}]} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}} \,. \tag{6.3}$$

Definition: A p-form $\omega \in \Omega^p(M)$ is called **closed** if it is in the kernel of d, i.e.

$$d\omega = 0\,, \tag{6.4}$$

and **exact** if it is the image of d, i.e. $\exists\, \chi \in \Omega^{p-1}(M)$ such that

$$\omega = d\chi\,. \tag{6.5}$$

We denote by

$$Z^p = \{\omega_p \mid \omega_p \in \Omega^p(M)\,,\ d\omega_p = 0\}\,, \tag{6.6}$$

the space of closed p-forms and by

$$B^p = \left\{\omega_p \mid \omega_p \in \Omega^p(M)\,,\ \exists\, \beta_{p-1} \in \Omega^{p-1} \text{ such that } \omega_p = d\beta_{p-1}\right\}\,, \tag{6.7}$$

the space of exact p-forms.

Definition: Let (M, g) be an n-dimensional oriented Riemannian manifold. Then we define the **Hodge dual** acting on p-forms as

$$\star : \Omega^p(M) \longrightarrow \Omega^{n-p}(M)\,. \tag{6.8}$$

For a p-form ω, with components in local coordinates as in (6.2), we define $\star$ as

$$\star\,\omega = \frac{1}{p!\,(n-p)\,!}\epsilon_{\mu_1\ldots\mu_p\mu_{p+1}\ldots\mu_n} g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p}\omega_{\nu_1\ldots\nu_p} dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_n}\,, \tag{6.9}$$

where $\epsilon_{\mu_1\ldots\mu_n}$ is the n-dimensional alternating tensor which takes values $\pm\sqrt{g}$ depending on whether $(\mu_1\ldots\mu_n)$ is an even or odd permutation of $(1\ldots n)$.

Note: The Hodge dual $\star$ depends on the metric whereas the exterior derivative d does not!

Exercise 6.1: Show that for a p-form ω

$$\star\star\omega = (-1)^{p(n-p)}\,\omega\,. \tag{6.10}$$

We can use the Hodge dual to define an inner product on p-forms.

Definition: Let (M, g) be an oriented Riemannian manifold. Then the **inner product** on p-forms,

$$(\, , \,) : \Omega^p(M) \otimes \Omega^p(M) \longrightarrow \mathbb{R}\,, \tag{6.11}$$

is defined as

$$(\alpha, \beta) = \int_M \alpha \wedge \star\beta \;\; \forall\, \alpha, \beta \in \Omega^p(M)\ . \tag{6.12}$$

Theorem 6.1: Let (M, g) be an oriented Riemannian manifold. The inner product for p-forms as defined above is symmetric and positive-definite.

Proof: It is easy to show that for p-forms $\alpha, \beta \in \Omega^p(M)$ the inner product is

$$(\alpha, \beta) = \int_M \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} \beta^{\mu_1 \ldots \mu_p} \sqrt{g} d^n x\,, \tag{6.13}$$

where $\beta^{\mu_1 \cdots \mu_p}$ are the components of β raised with the inverse metric. It is now clear that the inner product is symmetric. We can also see that

$$(\alpha, \alpha) = \int_M \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} \alpha^{\mu_1 \ldots \mu_p} \sqrt{g} d^n x > 0 \;\forall\, \alpha \neq 0 \tag{6.14}$$

is positive-definite because the metric g is positive-definite.

Exercise 6.2: Show that

$$\alpha_p \wedge \star\beta_p = \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} \beta^{\mu_1 \ldots \mu_p} \sqrt{g} dx^1 \wedge \ldots \wedge dx^n\,. \tag{6.15}$$

Since we now have a symmetric inner product on the vector space of p-forms, we can define adjoints of any operator. In particular, we define the adjoint of the exterior derivative $d^\dagger$, called the codifferential:

Definition: Let (M, g) be an oriented Riemannian manifold. The **codifferential**

$$d^\dagger : \Omega^p(M) \longrightarrow \Omega^{p-1}(M) \tag{6.16}$$

is defined for any $\alpha_p \in \Omega^p(M)$ and $\beta_{p-1} \in \Omega^{p-1}(M)$ as

$$(\alpha_p, d\beta_{p-1}) = \left(d^\dagger \alpha_p, \beta_{p-1}\right)\ . \tag{6.17}$$

Let us from now onwards assume that we have a compact manifold without boundary, $\partial M = 0$. We call manifolds without boundary closed. This implies that the codifferential does not contain boundary terms.

Theorem 6.2: Let (M, g) be an n-dimensional oriented, compact, closed Riemannian manifold. Then the codifferential is given by

$$d^\dagger = (-1)^{pn-n+1} \star d \star \; . \tag{6.18}$$

Proof: Let us begin by using the fact that the inner product is symmetric so that

$$(\alpha_p, d\beta_{p-1}) = \int_M d\beta_{p-1} \wedge \star\alpha_p \, , \tag{6.19}$$

and integrate by parts

$$(\alpha_p, d\beta_{p-1}) = -\,(-1)^{p-1} \int_M \beta_{p-1} \wedge d \star \alpha_p \, . \tag{6.20}$$

Now use the fact that $d \star \alpha_p$ is a $n - p + 1$ form and so

$$\star \star d \star \alpha = (-1)^{(n-p+1)(p-1)} \, d \star \alpha \, . \tag{6.21}$$

Thus, we can write (6.20) as

$$(\alpha_p, d\beta_{p-1}) = (-1)^{(n-p+1)(p-1)+p} \int_M \beta_{p-1} \wedge \star \left(\star d \star \alpha_p\right) \, . \tag{6.22}$$

Now, let us simplify the exponent of -1 by noticing that $p\,(p-1)$ is always even. Finally comparing to the definition of the codifferential (6.17) we find

$$d^\dagger = (-1)^{pn-n+1} \star d \star \; . \tag{6.23}$$

This completes the proof.
Corollary: The codifferential $d^\dagger$ is nilpotent, i.e. $d^\dagger d^\dagger = 0$.
Proof: This follows from the definition. For a p-form:

$$d^\dagger d^\dagger = (-1)^n \star d \star \star d \star = (-1)^{p(n-p)+n} \star d^2 \star = 0 \, , \tag{6.24}$$

because $d^2 = 0$.

Exercise 6.3: Write down the explicit expression for $d^\dagger$ acting on p-forms in 3 dimensions.

Exercise 6.4: Show that for a p-form $\omega \in \Omega^p$ with components

$$\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \,, \tag{6.25}$$

the codifferential acts as

$$d^\dagger \omega = -\frac{1}{(p-1)\,!}\nabla^\sigma \omega_{\sigma\mu_2\ldots\mu_p} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \,. \tag{6.26}$$

Hint: Recall that $\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}V^\mu\right) = \nabla_\mu V^\mu$.

Definition: A p-form $\omega \in \Omega^p\,(M)$ is called **co-closed** if it is in the kernel of $d^\dagger$, i.e.

$$d^\dagger \omega = 0\,, \tag{6.27}$$

and **co-exact** if it is the image of $d^\dagger$, i.e. $\exists\, \chi \in \Omega^{p+1}\,(M)$ such that

$$\omega = d^\dagger \chi\,. \tag{6.28}$$

We will denote by

$$\bar{Z}^p = \left\{\omega_p \mid \omega_p \in \Omega^p\,(M)\;,\; d^\dagger\omega_p = 0\right\}\,, \tag{6.29}$$

the space of co-closed p-forms and by

$$\bar{B}^p = \left\{\omega_p \mid \omega_p \in \Omega^p\,(M)\;,\; \exists \beta_{p+1} \in \Omega^{p+1} \text{ such that } \omega_p = d^\dagger\beta_{p+1}\right\}\,, \tag{6.30}$$

the space of exact p-forms.

We now have an operator that raises the rank of a p-form and one that lowers it. Thus, we can define an operator that takes p-forms to p-forms. This generalises our notion of the Laplacian operators acting on functions.

Definition: Let (M, g) be a compact, closed Riemannian manifold. Then we define the **Hodge-deRham operator**

$$\Delta : \Omega^p\,(M) \longrightarrow \Omega^p\,(M) \tag{6.31}$$

by

$$\Delta = dd^\dagger + d^\dagger d\,. \tag{6.32}$$

Definition: We call a p-form $\omega \in \Omega^p(M)$ a **harmonic** p-form if it lies in the kernel of Δ, i.e.

$$\Delta\omega = 0\,. \tag{6.33}$$

We label the vector space of harmonic p-forms as $\mathcal{H}^p(M)$.

Exercise 6.5: Write down the explicit expression of the Hodge-deRham operator acting on p-forms in 3 dimensions.

Exercise 6.6: Show that for a p-form $\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}$ the Hodge operator acts as

$$\Delta\omega = -\left(\frac{1}{p!}\nabla^\sigma\nabla_\sigma\omega_{\mu_1\ldots\mu_p} + \frac{1}{(p-1)\,!}\left[\nabla_{\mu_1},\nabla_\nu\right]\omega^\nu{}_{\mu_2\ldots\mu_p}\right)dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}\,, \tag{6.34}$$

where $\omega^\nu{}_{\mu_2\ldots\mu_p} = g^{\sigma\nu}\omega_{\sigma\mu_2\ldots\mu_p}$.

Exercise 6.7: Show that

$$\left[\nabla_\mu,\nabla_\nu\right]\omega^\nu{}_{\mu_2\ldots\mu_p} = -R_{\sigma\mu}\omega^\sigma{}_{\mu_2\ldots\mu_p} - \omega^\nu{}_{\sigma[\mu_3\ldots\mu_p}R^\sigma{}_{\mu_2]\mu\nu}\,. \tag{6.35}$$

Hint: Use normal coordinates so that the connection vanishes but the derivatives of the connection are non-vanishing.

Exercise 6.8: Use the result of the previous two exercises to show that

$$\begin{aligned}\Delta\omega = -\Bigg(&\frac{1}{p!}\nabla^\sigma\nabla_\sigma\omega_{\mu_1\ldots\mu_p} - \frac{1}{(p-1)\,!}R_{\sigma[\mu_1}\omega^\sigma{}_{\mu_2\ldots\mu_p]}\\ &+\frac{1}{(p-2)\,!}\omega^\nu{}_{\sigma[\mu_3\ldots\mu_p}R^\sigma{}_{\mu_1\mu_2]\nu}\Bigg)dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}\,.\end{aligned} \tag{6.36}$$

Theorem 6.3: A harmonic form is both closed and co-closed.

Proof: A harmonic form α satisfies

$$\Delta\alpha = dd^\dagger\alpha + d^\dagger d\alpha = 0\,. \tag{6.37}$$

Take the inner product of the above with α:

$$\left(\alpha, dd^\dagger\alpha\right) + \left(\alpha, d^\dagger d\alpha\right) = \left(d\alpha, d\alpha\right) + \left(d^\dagger\alpha, d^\dagger\alpha\right) = 0\,. \tag{6.38}$$

But since the inner product is positive-definite, the two terms must vanish independently. Thus we have

$$(d\alpha, d\alpha) = 0\,, \qquad \left(d^\dagger\alpha, d^\dagger\alpha\right) = 0\,. \tag{6.39}$$

But again because the inner product is positive-definite, this means that

$$d\alpha = 0\,, \qquad d^\dagger\alpha = 0\,, \tag{6.40}$$

and so α is both closed and co-closed. This completes the proof.

(Hodge) Theorem 6.4: Let (M, g) be a compact, closed Riemannian manifold. Then any p-form $\omega \in \Omega^p(M)$ has a *unique* decomposition into a harmonic, exact and co-exact part, i.e.

$$\omega = \alpha + d\beta + d^\dagger\gamma\,, \tag{6.41}$$

for some $\beta \in \Omega^{p-1}(M)$, $\gamma \in \Omega^{p+1}(M)$ and $\alpha \in \mathcal{H}^p(M)$. This is called the **Hodge decomposition**.

Proof: We will not prove the full theorem but we will show that the decomposition is unique. Assume that there are two different decompositions

$$\omega = \alpha + d\beta + d^\dagger\gamma = \alpha' + d\beta' + d^\dagger\gamma'\,, \tag{6.42}$$

with $\alpha, \alpha' \in \mathcal{H}^p(M)$. Let us denote by

$$\tilde{\alpha} = \alpha' - \alpha\,, \qquad \tilde{\beta} = \beta' - \beta\,, \qquad \tilde{\gamma} = \gamma' - \gamma\,. \tag{6.43}$$

Then we have that

$$\tilde{\alpha} + d\tilde{\beta} + d^\dagger\tilde{\gamma} = 0\,. \tag{6.44}$$

It remains to show that each term vanishes separately. To do so, act with d on the above equation to obtain

$$dd^\dagger\tilde{\gamma} = 0\,. \tag{6.45}$$

Taking the inner product of this with $\tilde{\gamma}$ we find

$$\left(\tilde{\gamma}, dd^\dagger\tilde{\gamma}\right) = \left(d^\dagger\tilde{\gamma}, d^\dagger\tilde{\gamma}\right) = 0\,. \tag{6.46}$$

Because the inner product is positive-definite this implies

$$d^\dagger\tilde{\gamma} = 0\,. \tag{6.47}$$

We can repeat this argument to find

$$d\tilde{\beta} = 0\,, \tag{6.48}$$

and hence $\tilde{\alpha} = 0$. Thus, the two decompositions are the same. This completes the proof.

Note: The Hodge decomposition implies that we can decompose the vector space of p-forms as

$$\Omega^p(M) = \mathcal{H}^p(M) \oplus B^p(M) \oplus \bar{B}^p(M)\,. \tag{6.49}$$

We will from now onwards often drop the argument (M).
Corollary: A closed form $\omega_p \in B^p$ can always be written as

$$\omega_p = \alpha + d\beta\,, \tag{6.50}$$

where $\alpha \in \mathcal{H}^p$ is harmonic, and similarly for a co-closed form.

Exercise 6.9: Prove the above, i.e. that a closed p-form can always be written as the sum of a harmonic and an exact p-form.

Let us now turn to the question of when we can write a closed p-form as an exact p-form. This question is the subject of deRham cohomology and is tied to the topology of the manifold as we will shortly see explicitly. Let us first state the Poincaré Lemma again.

(Poincaré) Theorem 6.5: Given a closed p-form ω,

$$d\omega = 0\,, \tag{6.51}$$

we can locally write $\omega = d\chi$ for some $(p-1)$-form $\chi \in \Omega^{p-1}(M)$.

This stems from the fact that locally any manifold looks like $\mathbb{R}^n$ and on $\mathbb{R}^n$ a closed p-form is an exact p-form. However, global properties of the manifold could mean that the χ defined is not globally well-defined.
Corollary: $B^p \subset Z^p$.
Proof: This follows from the fact that every exact form is closed.

Definition: The p-th **deRham cohomology group** is defined as

$$H^p = \frac{Z^p}{B^p}\,. \tag{6.52}$$

This means that H^p is the space of closed p-forms where we identify any two closed p-forms which differ by an exact p-form:

$$\omega_p \sim \omega_p + d\beta_{p-1}\ \forall\, \beta_{p-1} \in \Omega^{p-1}\,. \tag{6.53}$$

Given some closed p-form $\omega \in \Omega^p$ we define its equivalence class, the **cohomology class**

$$[\omega] \in H^p\,, \tag{6.54}$$

as the space of closed p-forms which differ from ω by an exact p-form. ω is called a **representative** of the cohomology class.

Note: The group operation for the cohomology group is addition.

The cohomology group is manifestly independent of the metric since it is only defined using the exterior derivative which does not require a metric. Thus, it is a topological property of the manifold. The space of harmonic forms, on the other hand, clearly depends on the metric since $d^\dagger$ is defined with respect to a metric. Thus, it may seem like these two vector spaces measure very different things. However, this is not the case as the following theorem shows.

Theorem 6.6: The space of harmonic p-forms $\mathcal{H}^p$ and the p-th cohomology group H^p are isomorphic.

Proof: We saw previously that a closed form can always be written as the unique sum of a harmonic and exact form. This defines the isomorphism.
Corollary: Every cohomology class contains a unique harmonic representative.

(Poincaré duality) Theorem 6.7: A p-form ω is harmonic if and only if $\star\omega$ is harmonic.

Proof: A p-form ω harmonic if and only if $d\omega = d^\dagger\omega = 0$. Consider now

$$d^\dagger \star \omega = (-1)^c \star d\omega\,, \tag{6.55}$$

and

$$d \star \omega = (-1)^{c'} \star d^\dagger \omega\,, \tag{6.56}$$

where c and c' are some integers (the factors of (-1) are unimportant here). Recall that $\star\star =$

$(-1)^{p(n-p)}$ and hence $\star$ is invertible. Thus, we see that

$$d^\dagger\omega = 0 \Longleftrightarrow d\star\omega = 0\,, \qquad d\omega = 0 \Longleftrightarrow d^\dagger\star\omega = 0\,. \tag{6.57}$$

This completes the proof.

Corollary: H^p and H^{n-p} are isomorphic.
Proof: This follows from the above together with the fact that the p-th deRham cohomology group and space of harmonic p-forms is isomorphic: $H^p \simeq \mathcal{H}^p$.

Example 6.1: H^0 is the space of constant functions on the manifold. This is because 0-forms are functions and there are no (-1)-forms hence no "exact functions". Thus, on a connected manifold we have $H^0 = \mathbb{R}$. If we have more than one connected component of the manifold then we can define a constant function on each as a generating element and so we have $H^0 = \mathbb{R}^c$ where c is the number of connected components.

Example 6.2: On a n-dimensional manifold, an n-forms is always closed and so H^n is generated by the volume form, if it exists. On an orientable manifold, there is a globally well-defined volume form which generates $H^n = \mathbb{R}$ whereas for a non-orientable manifold $H^n = 0$.

Example 6.3: $H^2\left(T^2\right) = \mathbb{R}^2$. Let us label the coordinates on the T^2 by coordinates x and y, subject to the identifications $x \sim x+1$, $y \sim y+1$. There are only two closed 1-forms which are not exact: dx and dy. Despite their suggestive form, dx and dy are not exact because the functions x and y do not respect the torus identifications and thus are not globally well-defined.

Note: You might wonder what happens to Poincaré duality in the example above when the manifold is not oriented. Clearly there are connected not-oriented manifolds (e.g. the Möbius strip). In this case the Hodge dual is not well-defined and so Poincaré duality does not hold.

From the two examples above we see that the dimensions of the cohomology groups are important. For example, they count the number of connected components of the manifold, or indicate whether the manifold is orientable.

Definition: We define the **Betti numbers**

$$b_p = \dim H^p\,, \tag{6.58}$$

to be the dimension of the cohomology groups.

Theorem 6.8: For an oriented, compact, closed Riemannian manifold, $b_p = b_{n-p}$.

Proof: This follows from Poincaré duality.

Definition: The **Euler characteristic** of a manifold is defined as the alternating sum of the Betti numbers:

$$\chi = \sum_{p=0}^{n} (-1)^p \, b^p \, . \tag{6.59}$$

Corollary: The Euler characteristic vanishes for odd-dimensional manifolds.
Proof: This follows immediately from the identity $b_p = b_{n-p}$.

It should be clear that the existence of closed but not exact forms (and hence harmonic forms) is closely related to the topology of the manifold. Let us take a simple example to get a better feel for what exactly this is due to. Let us consider a closed 1-form ω which can thus locally be written as

$$\omega = d\chi \, , \tag{6.60}$$

for some function χ. In $\mathbb{R}^n$ we could then construct χ by taking

$$\chi(x) = \int_y^x \omega(\xi)_\mu d\xi^\mu \, . \tag{6.61}$$

In $\mathbb{R}^n$ this is a valid construction since $\chi(x)$ is independent of the path chosen between y and x. (Changing y just corresponds to shifting χ by a constant which is of course going to leave ω invariant). This is because for two different paths γ_1 and γ_2 from y to x the difference in the definition of χ is just

$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\partial U} \omega = \int_U d\omega = 0 \, , \tag{6.62}$$

where ∂U is the boundary of the region U enclosed by the two curves γ_1, γ_2. Thus, the χ obtained by this procedure is well-defined. This is not true for a general manifold, as we can see by looking at the torus. For any two points, different paths connecting them are not in general the boundary of a region.

We thus see that the existence of harmonic forms and hence closed but not exact forms is related to the existence of closed curves that are not boundaries of a space. To understand this we need to study homology.

6.2 Homology

We will be somewhat brief in this section and not prove theorems to the same rigour as before. This is simply not doable within the scope of this course.

Definition: Given a manifold M, a **p-chain** a_p is a formal sum of smooth oriented p-dimensional submanifolds of M, call them N, so that

$$a_p = \sum_i r_i N_i \,, \tag{6.63}$$

where r_i are just some coefficients. When $r \in \mathbb{R}$ we have a **real p-chain**, whereas when $r \in \mathbb{C}$ we have a **complex p-chain**. Similarly, we can define p-chains over an arbitrary field K.

Note: Here we will only consider real p-chains.

Because we are using oriented p-dimensional submanifolds we can integrate p-forms over these. Thus, we can think of p-chains as something we can integrate a p-form over and the coefficients r_i then just define the weight of the various integrals over N_i:

$$\int_{\sum_i r_i N_i} = \sum_i r_i \int_{N_i} \,. \tag{6.64}$$

Definition: The space of p-chains is a vector space labelled by C_p.

Definition: The **boundary operator** ∂ associates to each manifold M its boundary ∂M. It maps a manifold of dimensions p to a manifold of dimension $p-1$.

Theorem 6.9:

$$\partial^2 = 0 \,. \tag{6.65}$$

Proof: The boundary of a boundary vanishes and so $\partial^2 M = 0$ for all M, i.e. $\partial^2 = 0$. This can be proven rigorously by "triangulating" your manifolds. This is called singular homology and is beyond the scope of these lectures.

Definition: We define the **boundary operator** to act on p-chains by linearity:

$$\partial \sum_i r_i N_i = \sum_i r_i \partial N_i \,. \tag{6.66}$$

Thus,

$$\partial : C_p \longrightarrow C_{p-1} \,. \tag{6.67}$$

Definition: A **p-cycle** is a p-chain without boundary, i.e.

$$\partial z_p = 0\,, \tag{6.68}$$

for all p-cycles z_p.

Definition: Let Z_p be the space of p-cycles and B_p the space of p-chains that are boundaries of $(p+1)$-chains:

$$B_p = \{a_p \in C_p \mid a_p = \partial a_{p+1} \text{ for some } a_{p+1} \in C_{p+1}\}\ . \tag{6.69}$$

Just as for differential forms we can now ask which cycles are not boundaries themselves. We do this by defining the homology group.

Definition: The **p-th homology group** is

$$H_p = \frac{Z_p}{B_p}\,. \tag{6.70}$$

Thus, H_p is the set of p-cycles with two cycles deemed equivalent if they differ by a boundary

$$a_p \sim a_p + \partial c_{p+1}\,. \tag{6.71}$$

Example 6.4: All points are 0-cycles since they have no boundary and any two points are the boundary of a curve. On each connected component we identify all points by the equivalence relation and so $H_0 = \mathbb{R}^c$ where c is the number of connected components.

Example 6.5: Consider T^2. Any two cycles going around the torus in the "same direction" are homologous, such as a_1 and a_2 in 1a, or trivial if they do not wrap the "hole" of the T^2 and are therefore a boundary, such as b' in 1a. Therefore there are only two homologically distinct 1-cycles, the a and b cycles shown in figure 1b.

Example 6.6: The homology groups of T^2: $H_0 = \mathbb{R}$ since the torus is connected. H^1 is generated by two different 1-cycles as we discussed above and so $H_1 = \mathbb{R}^2$. Finally, the only 2-chain without boundary is the T^2 itself and so we have $H_2 = \mathbb{R}$. You may wonder why the T^2 is not the boundary of the space it encloses (just as a sphere is the boundary of the ball it encloses). The reason is that the enclosed space is itself not part of the T^2 manifold.

Note: The fact that the homology groups here agree with the cohomology group is not an accident. We will see that this is true in general.

Example 6.7: The homology groups of S^n: $H_0 = \mathbb{R}$ since the n-sphere is connected. $H^k = 0$

(a) $a_1 \sim a_2$ since $a_1 - a_2$ is the boundary of the shaded region enclosed. b' is a trivial 1-cycle as it is the boundary of the shaded space enclosed.

(b) The a and b cycles are the only homologously distinct 1-cycles.

Figure 1: T^2 has only two homologously distinct 1-cycles a and b, as shown in (a). Any other 1-cycles are either a boundary or differ from a or b by a boundary, as shown in (b).

$\forall\ 0 < k < n$ because each (hyper-)circle is the boundary of some half-sphere. $H^n = \mathbb{R}$ since the n-sphere has no boundary and is itself not a boundary.

6.3 deRam's theorems and the isomorphism between homology and cohomology

Given a closed p-form and a p-cycle, it is natural to integrate one over the other. This defines a natural inner product between Z^p and Z_p.

Definition: Given a closed p-form $\omega_p \in Z^p$ and a p-cycle $a_p \in Z_p$, we define the **period** of ω_p over a_p as

$$\pi\left(a_p, \omega_p\right) = \int_{a_p} \omega_p\,. \qquad (6.72)$$

Theorem 6.10: The period function π defined above is a function on homology and cohomology classes, i.e.

$$\pi : H_p \otimes H^p \longrightarrow \mathbb{R}\,. \qquad (6.73)$$

Proof: The period function as defined in (6.72) above looks like a map from $Z_p \otimes Z^p \longrightarrow \mathbb{R}$. We wish to show that the period of any element of a cohomology class over any cycles separated by a boundary is the same. Consider thus the closed p-form $\omega'_p = \omega_p + d\beta_{p-1}$ and p-cycle

$a'_p = a_p + \delta c_{p+1}$. Then using Stoke's theorem we find that the period is

$$\begin{aligned}
\pi(a'_p, \omega'_p) &= \int_{a_p + \delta c_{p+1}} (\omega_p + d\beta_{p-1}) \\
&= \int_{a_p} \omega_p + \int_{a_p} d\beta_{p-1} + \int_{\delta c_{p+1}} \omega_p + \int_{\delta c_{p+1}} d\beta_{p-1} \\
&= \int_{a_p} \omega_p + \int_{\delta a_p} \beta_{p-1} + \int_{c_{p+1}} d\omega_p + \int_{c_{p+1}} d^2\beta_{p-1} \\
&= \int_{a_p} \omega_p \,.
\end{aligned} \tag{6.74}$$

Thus the period evaluated on different representatives of the same (co-)homology class is the same.

There are two important theorems involving the period, known as deRham's theorems, which together show that the p-th cohomology and p-th homology groups are isomorphic to each other. We will only state these theorems here.

(deRham's 1st) Theorem 6.11: Let $\{z_i\}$ be a basis for H_p. Then given any set of numbers α_i, $i = 1, \ldots \dim(H_p)$, there exists a closed p-form $\omega \in Z^p$ such that $\pi(z_i, \omega) = \alpha_i$.

(deRham's 2nd) Theorem 6.12: If all the periods of a closed p-form $\omega \in Z^p$ vanish then ω is exact.

If $\{z_i\}$ is a basis for H_p and $\{\omega^i\}$ is a basis for H^p then the period matrix

$$\pi_i{}^j = \pi\left(z_i, \omega^j\right) \tag{6.75}$$

is invertible and thus H_p and H^p are isomorphic. The isomorphism can be made more concrete using Poincaré duality (and recalling that $H^p \simeq H^{n-p}$).

(Poincaré Duality) Theorem 6.13: Given any p-cycle $a \in Z_p$ there exists an $(n-p)$-form α, the so-called **Poincaré dual** of a, such that for any closed p-form $\omega \in Z^p$

$$\int_a \omega = \int_M \alpha \wedge \omega \,. \tag{6.76}$$