Advanced topics in string and field theory: Complex manifolds and Calabi-Yau manifolds

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8 Calabi-Yau manifolds

We are now finally in a position to discuss Calabi-Yau manifolds. We will begin by defining these. There are in fact several different equivalent definitions which we hope to understand by the end of this chapter. We will conclude with some examples.

8.1 Yau's Theorem

Let us begin by expanding on what we have just learnt in the previous chapter.

Theorem 8.1: If a Kähler manifold admits a Ricci-flat metric its first Chern class vanishes.

Proof: Let g be the Ricci-flat metric and let us assume the Kähler manifold admits a different, not necessarily Ricci-flat, metric g'. Then by Theorem 7.10 we know that the Ricci-form of the two metrics is related by

$$R(g') = R(g) - d\mathcal{A} . \tag{8.1}$$

Since $R(g) = 0$ we see that $R(g')$ vanishes in the cohomology class and hence $c_1 = 0$.

Corollary: $\mathbb{CP}^N$ does not admit a Ricci-flat metric.
Proof: Recall from exercise 5.8 that on $\mathbb{CP}^N$ the Ricci form is given by

$$\mathcal{R} = -(N+1)\,\omega , \tag{8.2}$$

where ω is the Kähler form. We saw in chapter 7 that the Kähler form is harmonic. This implies it is not exact and neither is $\mathcal{R}$. Hence $c_1 \neq 0$ and there cannot be a Ricci-flat metric.

We see that c_1 is a topological obstruction to having a Ricci-flat metric on a Kähler manifold. Calabi conjectured and Yau proved that this is the only topological obstruction.

(Yau's) Theorem 8.2: Given a Kähler manifold (M, g, ω), let $\mathcal{R}$ be any $(1,1)$-form representing the first Chern class. Then there exists a unique metric g' on M with Kähler form ω' in the same Kähler class as ω whose Ricci-form is $\mathcal{R}$.

We will not prove the theorem here. However, notice that as a special case of the theorem we find the following.

Corollary: Let (M, g, ω) be a Kähler manifold with $c_1 = 0$. Then there exists a unique Ricci-flat metric g' on M with Kähler form ω' in the same Kähler class as ω.

This allows us to define a Calabi-Yau manifold.

Definition: A **Calabi-Yau manifold** is a compact Kähler manifold with vanishing first Chern class.

Theorem 8.3: A $2n$-dimensional Calabi-Yau manifold is a Kähler manifold with the following equivalent properties:

- It has vanishing first Chern class.
- It admits a Ricci-flat metric.
- It admits a metric whose holonomy group is (a subgroup of) $SU(N)$.
- There is a unique (up to a constant) holomorphic and nowhere vanishing $(n,0)$-form Ω.
- It admits a pair of globally well-defined covariantly constant spinors.

We have already shown that a Ricci-flat metric implies $c_1 = 0$ and stated the converse in the form of Yau's Theorem. We have proven the equivalence of $SU(N)$ holonomy and a Ricci-flat metric in chapter 5. Here we will only prove that a holomorphic and nowhere vanishing $(n,0)$-form implies Calabi-Yau although the other equivalences are discussed in Candelas' lecture notes as well as partially in Green, Schwarz, Witten.

Theorem 8.4: A compact $2n$-dimensional Kähler manifold with a nowhere vanishing holomorphic $(n,0)$-form Ω is Calabi-Yau.

Proof: An $(n,0)$-form $\Omega \in \Omega^{(n,0)}$ has to be proportional to the $n-$dimensional alternating symbol $\eta_{a_1 \ldots a_n} = \frac{1}{\sqrt{g}} \epsilon_{a_1 \ldots a_n}$. Since the $(n,0)$-form is holomorphic we can write

$$\Omega_{a_1 \ldots a_n} = f(z) \eta_{a_1 \ldots a_n} \, . \tag{8.3}$$

Using a Hermitian metric on the Kähler manifold, we can form the globally well-defined coordinate scalar

$$||\Omega||^2 = \frac{1}{n!}\Omega_{a_1\dots a_n} g^{a_1\bar{b}_1}\dots g^{a_n\bar{b}_n}\bar{\Omega}_{\bar{b}_1\dots\bar{b}_n} = \frac{|f|^2}{\sqrt{g}} \,. \tag{8.4}$$

Inverting this expression we find

$$\sqrt{g} = \frac{|f|^2}{||\Omega||^2} \,. \tag{8.5}$$

Recall that the Ricci-form is given by

$$\mathcal{R} = -i\partial\bar{\partial}\ln\sqrt{g} \,. \tag{8.6}$$

Plugging in (8.5) into this formula we see that

$$\mathcal{R} = i\partial\bar{\partial}\ln||\Omega||^2 \,. \tag{8.7}$$

But as we said $||\Omega||^2$ is a globally well-defined coordinate scalar and thus $\mathcal{R}$ is exact. This implies $c_1 = 0$ and completes the proof.

This is a very important theorem since it is often easy to find Calabi-Yau manifolds by trying to construct a nowhere vanishing holomorphic $(n,0)$-form. We will see this later when we give examples of Calabi-Yaus.

We can also show that such a $(n,0)$-form is unique up to a constant. Let us begin with the following theorem.

Theorem 8.5: A nowhere vanishing holomorphic $(n,0)$-form Ω on a $2n$-dimensional Kähler manifold is harmonic.

Proof: Let us first show that $d\Omega = 0$.

$$d\Omega = \partial\Omega + \bar{\partial}\Omega \,. \tag{8.8}$$

Ω is a $(n,0)$-form and so

$$\partial\Omega = 0 \,. \tag{8.9}$$

By assumption Ω is holomorphic and thus

$$\bar{\partial}\Omega = 0 \,. \tag{8.10}$$

Now let's turn to $d^\dagger\Omega$. In local coordinates this is

$$\begin{aligned} d^\dagger\Omega &= -\frac{1}{(n-1)\,!}\nabla^a\Omega_{ab_1\dots b_{n-1}} dz^{b_1}\wedge\dots\wedge dz^{b_{n-1}} \\ &= -\frac{1}{(n-1)\,!} g^{a\bar{c}}\nabla_{\bar{c}}\Omega_{ab_1\dots b_{n-1}} dz^{b_1}\wedge\dots\wedge dz^{b_{n-1}} \,. \end{aligned} \tag{8.11}$$

Recall that the connection is pure in its indices so that the covariant derivative just reduces to the partial derivative

$$d^\dagger\Omega = -\frac{1}{(n-1)\,!}g^{a\bar{c}}\partial_{\bar{c}}\Omega_{ab_1\ldots b_{n-1}}dz^{b_1}\wedge\ldots\wedge dz^{b_{n-1}} = 0\,. \tag{8.12}$$

This completes the proof.

> **Theorem 8.6:** On a complex $2n$-dimensional complex manifold there is at most one unique (up to a constant) globally defined holomorphic $(n,0)$-form.

Proof: Let Ω and Ω' be two such globally defined holomorphic $(n,0)$-forms. They can only be proportional to the n-dimensional permutation symbol and hence must be proportional to each other:

$$\Omega' = h(z)\Omega\,, \tag{8.13}$$

where $h(z)$ must a globally defined holomorphic function. By the maximum modulus principle such a function must be a constant.

8.2 Hodge diamond of Calabi-Yau manifolds

We saw that the extra structure of Kähler manifolds imply certain relations on their hodge numbers. Let us now see what happens if we have a Calabi-Yau. Let us begin by stating a theorem we will not prove but that is useful.

> **Theorem 8.7:** On a manifold with Euler number χ any vector field has at least $|\chi|$ zeros.

Let us use this to prove the following result.

> **Theorem 8.8:** A Calabi-Yau manifold with Euler number $\chi \neq 0$ has $h^{1,0} = 0$.

Proof: Notice first of all that the first Betti number $b^1 = 2h^{10}$. Thus $b^1 = 0$ iff $h^{10} = 0$. This is useful because the Betti numbers are topological invariants whereas the Hodge numbers are not – they depend on the complex structure but not the metric. Because the Betti numbers are topological invariants we can consider the Ricci-flat metric for simplicity.

Let us now assume that $\omega \in \mathcal{H}^1$ is a harmonic 1-form:

$$\Delta\omega = 0\,. \tag{8.14}$$

Explicitly (using the result of exercise 6.8) we have

$$- \nabla^\nu \nabla_\nu \omega_\mu + R_\mu{}^\nu \omega_\nu = 0 \,. \tag{8.15}$$

As the Ricci tensor vanishes ω is only harmonic if

$$\nabla^\nu \nabla_\nu \omega_\mu = 0 \,. \tag{8.16}$$

Let us multiply this by ω^μ and integrate over the manifold M:

$$\int_M \sqrt{g}\omega^\mu \nabla^\nu \nabla_\nu \omega_\mu = - \int_M \sqrt{g} \nabla^\nu \omega^\mu \nabla_\nu \omega_\mu = 0 \,. \tag{8.17}$$

The integrand is positive definite and so we see that this implies $\nabla_\nu \omega_\mu = 0$. Using now the results of the previous theorem we see that ω must have at least one zero. However, if it is covariantly constant and vanishes at one point, then it vanishes everywhere and so $\omega = 0$.

This immediately implies the following.

Theorem 8.9: A $2n$-dimensional Calabi-Yau manifold has $h^{n,0} = 1$.

Proof: We saw that a Calabi-Yau manifold has a unique holomorphic harmonic $(n, 0)$-form. This implies $h^{n,0} \geq 1$. Let us now assume that there is a different harmonic $(n, 0)$-form Ω'. It has to be closed, i.e.

$$d\Omega' = \partial\Omega' + \bar{\partial}\Omega' = 0 \,. \tag{8.18}$$

But $\partial\Omega' = 0$ for a $(n, 0)$-form. Recall that any $(n, 0)$-form can be written as

$$\Omega' = f(z, \bar{z})\eta \,, \tag{8.19}$$

where η is the n-dimensional alternating symbol. Thus we find that

$$\bar{\partial} f(z, \bar{z}) = 0 \,, \tag{8.20}$$

and hence $f(z, \bar{z}) = f(z)$ is holomorphic. We have proven uniqueness of a holomorphic $(n, 0)$-form already and so this completes our proof.

Theorem 8.10: A $2n$-dimensional Calabi-Yau manifold has $h^{p,0} = h^{n-p,0}$.

Proof: Let $\omega_p \in \Omega^{p,0}$ be a $(p, 0)$-form. Then define a generalisation of a "Hodge dual" as:

$$v_{\bar{a}_1 \ldots \bar{a}_{n-p}} = \frac{1}{p!} \bar{\Omega}_{\bar{a}_1 \ldots \bar{a}_{n-p} \bar{a}_{n-p+1} \ldots \bar{a}_n} \omega^{\bar{a}_{n-p+1} \ldots \bar{a}_n} \,. \tag{8.21}$$

Taking the divergence of this expression we have (recall Ω is harmonic)

$$\nabla^{\bar{a}_1} v_{\bar{a}_1 \ldots \bar{a}_{n-p}} = \frac{1}{p!} \bar{\Omega}_{\bar{a}_1 \ldots \bar{a}_n} \nabla^{\bar{a}_1} \omega^{\bar{a}_{n-p+1} \ldots \bar{a}_n} \,. \tag{8.22}$$

Let us also invert (8.21) to find

$$\omega^{\bar{a}_{n-p+1} \ldots \bar{a}_n} = \frac{n!}{(n-p)\,!\,p!} \frac{1}{||\Omega||^2} \Omega^{\bar{a}_1 \ldots \bar{a}_n} v_{\bar{a}_1 \ldots \bar{a}_{n-p}} \,. \tag{8.23}$$

Taking the divergence of this we get

$$\nabla_{\bar{a}_n} \omega^{\bar{a}_{n-p+1} \ldots \bar{a}_n} = \frac{n!}{(n-p)\,!\,p!} \frac{\Omega^{\bar{a}_1 \ldots \bar{a}_n}}{||\Omega||^2} \nabla_{\bar{a}_n} v_{\bar{a}_1 \ldots \bar{a}_{n-p}} \,. \tag{8.24}$$

Because $\frac{\bar{\Omega}}{||\Omega||^2}$ is holomorphic and non-singular, the relations (8.22) and (8.24) imply that

$$dv = 0 \Longleftrightarrow d^\dagger \omega = 0 \,, \qquad d^\dagger v = 0 \Longleftrightarrow d\omega = 0 \,. \tag{8.25}$$

Thus, v is harmonic if and only if ω is harmonic and this completes the proof.

Exercise 8.1: Invert (8.21) to find (8.23).

Exercise 8.2: In the above proof we used the fact that $\bar{\Omega}/||\Omega||^2$ is holomorphic and non-singular. Prove this by writing $\Omega = f(z)\eta$ where $f(z)$ is a holomorphic function and η is the n-dimensional alternating symbol.

Let us now put all this together and study the Hodge diamonds of low-dimensional Calabi-Yau manifolds. Let us begin in two dimensions. There, we know that $h^{1,0} = 1$ because there must be a holomorphic 1-form. This together with the symmetries $h^{p,0} = h^{n-p,0}$ and $h^{p,q} = h^{n-p,n-q}$ is enough to fully determine the Hodge diamond:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \,. \tag{8.26}$$

From theorem 8.8 we see that $h^{1,0} \neq 0$ implies that $\chi = 0$. There are only two manifolds that have vanishing Euler number: the torus and Klein bottle. $h^{1,1} = 1$ tells us this Calabi-Yau must be a T^2 since the Klein bottle is non-orientable. Thus we see that the only 2-dimensional Calabi-Yau manifold is a torus!

Let us now turn to four dimensions and consider the case $\chi \neq 0$. The resulting Hodge diamond is

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & h^{1,1} & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array} \quad . \tag{8.27}
$$

We see that the Hodge diamond for four-dimensional Calabi-Yau manifolds with non-vanishing Euler characteristic only depends on one Hodge number $h^{1,1}$. These manifolds are called "K3 surfaces" and it can be proven that they are all diffeomorphic to each other. (T^4 is also a Calabi-Yau but has vanishing Euler characteristic.)

Finally, let us consider the physically relevant case of six-dimensional Calabi-Yau manifolds. The Hodge diamond is now determined by only two Hodge numbers:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{1,2} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \quad . \tag{8.28}
$$

Exercise 8.3: Show that for a six-dimensional Calabi-Yau with non-vanishing Euler characteristic, the Euler characteristic is given by

$$
\chi = 2\left(h^{1,1} - h1,2\right) \ . \tag{8.29}
$$

8.3 Examples of Calabi-Yau manifolds

We have already met T^2 and T^4 as examples of Calabi-Yau manifolds. There is currently no systematic way of constructing Calabi-Yau manifolds. However, there are some fruitful approaches, such as considering submanifolds of $\mathbb{CP}^N$ or smoothened orbifolds by "blowing up" their singularities. Here we will only consider the first approach. Note that it is very hard to find metrics on Calabi-Yau manifolds and generally we only know the manifold but not the metric!

Definition: An **analytic submanifold** of a manifold M is determined by the locus of holomorphic equations $F(z) = 0$ on M.

Theorem 8.11: An analytic submanifold of a Kähler manifold is also Kähler.

Proof: This follows from the fact that the restriction of the Kähler metric and Kähler form to an analytic submanifold gives itself a Kähler metric and Kähler form.

One may try and construct Calabi-Yau's from compact submanifolds of $\mathbb{C}^n$. For these to be complex they need to be analytic submanifolds. However, the only compact, connected holomorphic submanifolds of $\mathbb{C}^n$ are points. This comes from the maximum modulus principle.

We saw that $\mathbb{CP}^N$ is Kähler but not Calabi-Yau. We can now try and find analytic submanifolds of $\mathbb{CP}^N$ (which are thus all Kähler) which are Calabi-Yau. $\mathbb{CP}^N$ avoids the maximum modulus issue because it is compact.

There is a theorem by Chow that all analytic submanifolds of $\mathbb{CP}^N$ can be described as the locus of a finite number of holomorphic homogeneous polynomial equations. We will keep things simple here and consider only those submanifolds given by the locus of a single holomorphic homogeneous polynomial equation. The generalisation to the intersection of several polynomial equations is fairly simple and you can read about it in Candelas' lecture notes as well as in Green, Schwarz, Witten.

Example 8.1: Let us start by considering 1-dimensional submanifolds of $\mathbb{CP}^2$. Let us use the homogeneous coordinates of $\mathbb{C}^3$: z_1, z_2, z_3. In order to obtain a homogeneous polynomial each term has to have the same combined power of all z's. For simplicity let us consider the polynomial

$$P(z) = z_1^n + z_2^n + z_3^n \,. \tag{8.30}$$

The surface $P(z) = 0$ is well-defined in $\mathbb{CP}^2$ because $P(\lambda z) = \lambda^n P(z) = 0$ respects the equivalence relation used to obtain $\mathbb{CP}^2$ from $\mathbb{C}^3$. Let us find some values of n for which this is a Calabi-Yau by attempting to construct a holomorphic 1-form. Let us start by considering the patch $z_1 \neq 0$ and defining the inhomogeneous coordinates $x = z_2/z_1$ and $y = z_3/z_1$. We can then write

$$P(x, y) = z_1^n (1 + x^n + y^n) = z_1^n p(x, y) = 0 \,. \tag{8.31}$$

Let us define a 1-form in this patch as

$$\Omega_1 = \frac{dx}{\partial p/\partial y} \,. \tag{8.32}$$

On first sight this looks no good because it is singular whenever $\frac{\partial p}{\partial y} = 0$. However, recall that $p = 0$ on the surface we are considering and thus

$$dp = 0 = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \,. \tag{8.33}$$

This implies that we can write

$$\Omega_1 = -\frac{dy}{\partial p/\partial x} \,. \tag{8.34}$$

Now we see that Ω_1 is only non-singular when both $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$. However, at such a point $dp = 0$ and so the normal to the surface would vanish there. This implies the surface has a cusp and so we are happy to discard such surfaces.

Assuming now that we only consider homogeneous holomorphic polynomial equations such that $dp \neq 0$ everywhere, we need to consider the overlap between two different coordinate patches and show that Ω_1 is well-defined. Consider thus the region where $z_2 \neq 0$ and define there the inhomogeneous coordinates $\tilde{x} = z_1/z_2$, $\tilde{y} = z_3/z_2$. Now the polynomial is

$$P(\tilde{x}, \tilde{y}) = z_2^n \left(\tilde{x}^n + \tilde{y}^n + 1\right) = z_2^n \tilde{p}(\tilde{x}, \tilde{y}) = 0\,, \tag{8.35}$$

and define in this region the 1-form

$$\Omega_2 = -\frac{d\tilde{x}}{\partial \tilde{p}/\partial \tilde{y}}\,. \tag{8.36}$$

Let us evaluate the one-forms in the overlap where $z_1 \neq 0$, $z_2 \neq 0$. There we find

$$\Omega_1 = \frac{dx}{ny^{n-1}}\,, \qquad \Omega_2 = -\frac{d\tilde{x}}{n\tilde{y}^{n-1}}\,. \tag{8.37}$$

However, the coordinates are related by

$$\tilde{x} = x^{-1}\,, \qquad \tilde{y} = yx^{-1}\,, \tag{8.38}$$

and thus we see that

$$\Omega_2 = x^{n-3}\Omega_1\,. \tag{8.39}$$

We see that when $n = 3$ i.e. for a cubic polynomial, the 1-form considered is well-defined in the regions $z_1 \neq 0$ and $z_2 \neq 0$ and their overlap. We can continue this procedure to show that the one-form is globally well-defined. This is clearly a holomorphic $(1,0)$-form which vanishes nowhere and thus this manifold is Calabi-Yau.

Exercise 8.4: Construct a 1-form in the region $z_3 \neq 0$ which agrees with Ω_1 in the overlap $z_1 \neq 0$, $z_3 \neq 0$ and with Ω_2 in the overlap $z_2 \neq 0$, $z_3 \neq 0$.

It is a different matter to realise that we have just described a torus! We know this is the case from our discussion of Hodge diamonds. Note that a generic cubic homogeneous polynomial in $\mathbb{CP}^2$ can be written as

$$\sum_{i,j,k} a_{ijk} z_i z_j z_k = 0\,, \tag{8.40}$$

where i,j,$k = 1, 2, 3$. The (in general complex) coefficients a_{ijk} have to be completely symmetric in their indices and so there are $\frac{3\times4\times5}{3!} = 10$ independent complex parameters. However, any linear coordinate transformation on the z_i's (which corresponds to an element in $GL(3, \mathbb{C})$ leaves the form of the polynomial invariant. Thus we can fix 9 out of the 10 complex parameters in the polynomial and we are left with one independent complex parameter. This corresponds to the complex structure of the torus.

Example 8.2: Let us now construct $K3$ surfaces which are four-dimensional Calabi-Yau manifolds. We will proceed analogously to the two-dimensional case by considering a homogeneous polynomial in $\mathbb{CP}^3$. We will take the quartic polynomial in homogeneous coordinates on $\mathbb{C}^4$.

$$P(z) = z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\,. \tag{8.41}$$

Let us again consider the coordinate patch $z_1 \neq 0$ and introduce inhomogeneous coordinates

$$x = \frac{z_2}{z_1}\,, \qquad y = \frac{z_3}{z_1}\,, \qquad z = \frac{z_4}{z_1}\,. \tag{8.42}$$

The polynomial is then

$$P(x,y,z) = z_1^4 p(x,y,z) = z_1^4\left(x^4 + y^4 + z^4 + 1\right) = 0\,. \tag{8.43}$$

Let us define a two-form

$$\Omega_1 = \frac{dx \wedge dy}{\partial p/\partial z}\,. \tag{8.44}$$

Using

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z}dz = 0\,, \tag{8.45}$$

and wedging the expression with dy we get

$$\frac{\partial p}{\partial x}dx \wedge dy = -\frac{\partial p}{\partial z}dz \wedge dy\,. \tag{8.46}$$

Thus we can write

$$\Omega_1 = \frac{dy \wedge dz}{\partial p/\partial x} = \frac{dz \wedge dx}{\partial p/\partial y}\,, \tag{8.47}$$

and again we see that if $dp \neq 0$ everywhere this two-form is non-singular.

Exercise 8.5: Show that this one-form can be defined globally and thus that this submanifold is a Calabi-Yau.

Let us end this discussion by counting the number of complex structure moduli on $K3$ surfaces. A general quartic polynomial can be written as

$$\sum_{i,j,k,l} a_{ijkl} z^i z^j z^k z^l = 0\,, \tag{8.48}$$

where now i, j, k, $l = 1, \ldots 4$. Thus there are $\frac{4\times5\times6\times7}{4!} = 35$ independent complex a_{ijkl}. Again we can act with linear coordinate transformations (i.e. $GL(4,\mathbb{C})$ and leave the polynomial invariant.

This means we can fix 16 complex parameters. In total there are 19 independent complex parameters remaining which are the complex structure moduli.

Exercise 8.6: By considering the polynomial $P(z) = z_1^n + z_2^n + z_3^n + z_4^n = 0$ show that the above construction only works for $n = 4$.

Exercise 8.7: Generalise the construction above to find a holomorphic $(3, 0)$-form for a quintic polynomial in $\mathbb{CP}^4$ thus constructing a 6-dimensional Calabi-Yau manifold. Show that there are 101 complex structure moduli of these kinds of 6-dimensional Calabi-Yau manifolds.