Quantum Field Theory: Example Sheet 1

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1. A string of length a, mass per unit length σ and under tension T is fixed at each end. The Lagrangian governing the time evolution of small transverse displacements $y(x,t)$ is

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right]$$

where x identifies position along the string from one end point. By expressing the displacement as a sine series Fourier expansion of the form

$$y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} q_n(t) \, \sin\left(\frac{n\pi x}{a} \right)$$

show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] .$$

Derive the equations of motion. Hence show that the string is equivalent to an infinite set of decoupled harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a} \right) .$$

2. A string has classical Hamiltonian given by

$$H = \sum_{n=1}^{\infty} \left(\tfrac{1}{2} p_n^2 + \tfrac{1}{2} \omega_n^2 \, q_n^2 \right)$$

where ω_n is the frequency of the nth mode. Compare this Hamiltonian to the Lagrangian in the previous question. The mass per unit length, σ, has now been set to unity so as to make various formulae somewhat simpler.

After quantization, q_n and p_n become operators satisfying

$$[q_n, q_m] = [p_n, p_m] = 0 \quad \text{and} \quad [q_n, p_m] = i\delta_{nm} .$$

Introduce creation and annihilation operators a_n and $a_n^\dagger$,

$$a_n = \sqrt{\frac{\omega_n}{2}}\, q_n + \frac{i}{\sqrt{2\omega_n}}\, p_n \quad \text{and} \quad a_n^\dagger = \sqrt{\frac{\omega_n}{2}}\, q_n - \frac{i}{\sqrt{2\omega_n}}\, p_n\,.$$

Show that they satisfy the commutation relations

$$[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \quad \text{and} \quad [a_n, a_m^\dagger] = \delta_{nm}\,.$$

Show that the Hamiltonian of the system can be written in the form

$$H = \sum_{n=1}^{\infty} \tfrac{1}{2}\,\omega_n \left(a_n a_n^\dagger + a_n^\dagger a_n\right)\,.$$

Given the existence of a ground state $|0\rangle$ such that $a_n|0\rangle = 0$, explain how, after removing the vacuum energy, the Hamiltonian can be expressed as

$$H = \sum_{n=1}^{\infty} \omega_n a_n^\dagger a_n\,.$$

Show further that $[H, a_n^\dagger] = \omega_n\, a_n^\dagger$ and hence calculate the energy of the state

$$|l_1, l_2, \ldots, l_N\rangle = \left(a_1^\dagger\right)^{l_1} \left(a_2^\dagger\right)^{l_2} \ldots \left(a_N^\dagger\right)^{l_N} |0\rangle\,.$$

3. Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation Λ.

4. The motion of a complex field $\phi(x)$ is governed by the Lagrangian density

$$\mathcal{L} = -\partial_a\phi^*\partial^a\phi - m^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2\,.$$

Write down the Euler-Lagrange field equations for this system. Verify that the Lagrangian density is invariant under the infinitesimal transformation

$$\delta\phi = i\alpha\phi \quad , \quad \delta\phi^* = -i\alpha\phi^*\,.$$

Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equation satisfied by ϕ.

5. Verify that the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_a\phi_\alpha\partial^a\phi_\alpha - \frac{1}{2}m^2\phi_\alpha\phi_\alpha$$

for a triplet of real fields ϕ_α ($\alpha = 1, 2, 3$) is invariant under the infinitesimal $SO(3)$ rotation by θ

$$\phi_\alpha \to \phi_\alpha + \theta \epsilon_{\alpha\beta\gamma} n_\beta \phi_\gamma$$

where n_α is a constant unit vector. Compute the Noether current j^a. Deduce that the three quantities

$$Q_\alpha = \int d^3x \; \epsilon_{\alpha\beta\gamma} \, \dot{\phi}_\beta \phi_\gamma$$

are all conserved and verify this directly using the field equations satisfied by ϕ_α.

6. A Lorentz transformation $x^a \to x'^a = \Lambda^a{}_b x^b$ is such that it preserves the Minkowski metric η_{ab}, meaning that $\eta_{ab} x^a x^b = \eta_{ab} x'^a x'^b$ for all x. Show that this implies that

$$\eta_{ab} = \eta_{cd} \Lambda^c{}_a \Lambda^d{}_b \,. \qquad (*)$$

Use this result to show that an infinitesimal transformation of the form

$$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b$$

is a Lorentz transformation when ω^{ab} is antisymmetric: i.e. $\omega^{ab} = -\omega^{ba}$.

Write down the matrix form for $\omega^a{}_b$ that corresponds to a rotation through an infinitesimal angle θ about the x^3-axis. Do the same for a boost along the x^1-axis by an infinitesimal velocity v. Deduce the form of a Lorentz transformation for a finite rotation about the x^3-axis, and for a finite boost along the x^1-axis by calculating the exponential of the matrix you found. (Hint: Look carefully at the squares of the matrices.) Verify that your results are indeed Lorentz transformations by checking that they satisy (*)

7. Consider the infinitesimal form of the Lorentz transformation derived in the previous question: $x^a \to x^a + \omega^a{}_b x^b$. Show that a scalar field transforms as

$$\phi(x) \to \phi'(x) = \phi(x) - \omega^a{}_b \, x^b \, \partial_a \phi(x)$$

and hence show that the variation of the Lagrangian density is a total derivative

$$\delta \mathcal{L} = -\partial_a (\omega^a{}_b x^b \, \mathcal{L}) \,.$$

Using Noether's theorem deduce the existence of the conserved current

$$j^a = -\omega^c{}_b \, (T^a_c \, x^b) \,.$$

The three conserved charges arising from spatial rotational invariance define the *total angular momentum* of the field. Show that these charges are given by

$$Q_i = \epsilon_{ijk} \int d^3x \, \left(x^j T^{0k} - x^k T^{0j} \right) .$$

Derive the conserved charges arising from invariance under Lorentz boosts. Show that they imply

$$\frac{d}{dt} \int d^3x \, (x^i \, T^{00}) = \text{constant}$$

and interpret this equation.

8 A class of interesting theories are those invariant under the scaling of all lengths by

$$x^a \to x'^a = \lambda \, x^a \quad \text{and} \quad \phi(x) \to \phi'(x) = \lambda^{-D} \phi(\lambda^{-1} x) .$$

Here D is called the *scaling dimension* of the field. Consider the action for a real scalar field given by

$$S = \int d^4x \, \left(-\frac{1}{2} \, \partial_a \phi \partial^a \phi - \frac{1}{2} m^2 \phi^2 - g\phi^p \right) .$$

Find the scaling dimension D such that the part of S involving the derivative terms is invariant. For what values of m and p is the scaling (1) a symmetry of the theory. How do these conclusions change for a scalar field living in an $(n+1)$-dimensional spacetime instead of a $(3+1)$-dimensional spacetime?

In $3+1$ dimensions, use Noether's theorem to construct the conserved current D^a associated with scaling invariance.

9. Show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2 \phi(x) - m^2 \phi(x) .$$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation.

10. Let $\phi(x)$ be a real scalar field in the Heisenberg picture. Show that the relativistically normalized one-particle states $|p\rangle = \sqrt{2E_{\vec{p}}}\, a^\dagger \; |0\rangle$ satisfy

$$\langle 0| \, \phi(x) \, |p\rangle = e^{ip\cdot x} .$$

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