Quantum Field Theory: Example Sheet 2

Prof M.J. Perry, Halloween 2013.

1. Show that if $\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}$, then

$$\left[\gamma^a\gamma^b\, ,\, \gamma^c\gamma^d\right] = 2\eta^{bc}\gamma^a\gamma^d - 2\eta^{ac}\gamma^b\gamma^d + 2\eta^{bd}\gamma^c\gamma^a - 2\eta^{ad}\gamma^c\gamma^b\, .$$

By expressing $S^{ab} = \frac{1}{4}\left[\gamma^a\, ,\, \gamma^b\right]$ as $\frac{1}{2}(\gamma^a\gamma^b - \eta^{ab})$ etc., evaluate $\left[S^{ab}\, ,\, S^{cd}\right]$.

2. Show using the results of the previous question that if $S^i = -\frac{i}{4}\epsilon_{ijk}\gamma^j\gamma^k$, then $[S^i, S^j] = i\epsilon_{ijk}S^k$. Show also that $[\gamma^0, S^i] = [\gamma^5, S^i] = 0$, where $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that $(S^1)^2 = (S^2)^2 = (S^3)^2 = \frac{1}{4}$.

Verify these results in the representation used in the lectures. What can you deduce about rotations and spin in the Dirac field theory?

3. Using just the algebra $\{\gamma^a,\ \gamma^b\} = 2\eta^{ab}$ (i.e. without resorting to a particular representation), and defining $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} = p_a\gamma^a$ and $S^{ab} = \frac{1}{4}\left[\gamma^a,\ \gamma^b\right]$, prove the following results:

1. $\mathrm{Tr}\gamma^a = 0$
2. $\mathrm{Tr}(\gamma^a\gamma^b) = 4\eta^{ab}$
3. $\mathrm{Tr}(\gamma^a\gamma^b\gamma^c) = 0$
4. $(\gamma^5)^2 = 1$
5. $\mathrm{Tr}\gamma^5 = 0$
6. $\not{p}\,\not{q} = 2p\cdot q - \not{q}\,\not{p} = p\cdot q + 2S^{ab}p_a q_b$
7. $\mathrm{Tr}(\not{p}\,\not{q}) = 4p\cdot q$
8. $\mathrm{Tr}(\not{p}_1\ldots\not{p}_n) = 0$ if n is odd
9. $\mathrm{Tr}(\not{p}_1\,\not{p}_2\,\not{p}_3\,\not{p}_4) = 4\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4)\right]$
10. $\mathrm{Tr}(\gamma^5\,\not{p}_1\,\not{p}_2) = 0$
11. $\gamma_a\,\not{p}\,\gamma^a = -2\,\not{p}$

12. $\gamma_a \not{p}_1 \not{p}_2 \gamma^a = 4p_1 \cdot p_2$

13. $\gamma_\mu \not{p}_1 \not{p}_2 \not{p}_3 \gamma^\mu = -2 \not{p}_3 \not{p}_2 \not{p}_1$

14. $\mathrm{Tr}(\gamma^5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) = -4i\, \epsilon_{abcd}\, p_1^a\, p_2^b\, p_3^c\, p_4^d$

4. The Weyl representation of the Clifford algebra is

$$\gamma^0 = -i \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} . \tag{1}$$

Show that these indeed satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}$. Find a unitary matrix U such that $(\gamma')^a = U\gamma^a U^\dagger$, where $(\gamma')^a$ form the Dirac representation of the Clifford algebra

$$(\gamma')^0 = -i \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \quad , \quad (\gamma')^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} . \tag{2}$$

5. Starting with the Dirac equation, show that the Dirac conjugate field $\overline{\psi}(x)$ obeys

$$\partial_a \overline{\psi} \gamma^a - m\overline{\psi} = 0\,. \tag{3}$$

Derive this equation from the Dirac action as the variational equation with respect to ψ.

Show that $V^a = \overline{\psi}\gamma^a\psi$ is a Noether current, and hence conserved. Show that the axial vector currrent $A^a = \overline{\psi}\gamma^a\gamma^5\psi$ is conserved if $m = 0$.

6. Explain why one may split any Dirac spinor uniquely into left- and right-handed spinor parts $\psi = \psi_L + \psi_R$. Show that any gamma matrix γ^a maps a left-handed into a right-handed spinor, and vice versa, and deduce that any non-trivial solution of the Dirac equation with $m \neq 0$ has both left- and right-handed parts.

Find the plane wave solutions of the massless Dirac equation with purely a left-handed part. For a given value of the 3-momentum, what is the dimension of the space of solutions?

7. With the notation as in the lectures show that

$$\sum_{s=\pm\frac{1}{2}} u_s(\vec{p})\overline{u}_s(\vec{p}) \;=\; i\not{p} + m \tag{4}$$

$$\sum_{s=\pm\frac{1}{2}} v_s(\vec{p})\overline{v}_s(\vec{p}) \;=\; i\not{p} - m \tag{5}$$

where the two spinors on the left-hand side are placed back to back to form a 4×4 matrix.

8. The Fourier decomposition of the Dirac field operator in the Schrodinger representation is $\psi(\vec{x})$ and its conjugate momentum $\psi^\dagger(\vec{x})$ is given by

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=\pm\frac{1}{2}} \left[a^s_{\vec{p}} u_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^{s\,\dagger}_{\vec{p}} v_s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right]$$
$$\psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=\pm\frac{1}{2}} \left[a^{s\,\dagger}_{\vec{p}} u_s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + b^s_{\vec{p}} v_s(\vec{p})^\dagger e^{i\vec{p}\cdot\vec{x}} \right] . \tag{6}$$

The creation and annihilation operators are taken to satisfy

$$\begin{aligned} \{a^r_{\vec{p}}, a^{s\,\dagger}_{\vec{q}}\} &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q}) \\ \{b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\} &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q}) \,, \end{aligned} \tag{7}$$

with all other anticommutators vanishing; that is,

$$\{a^r_{\vec{p}}, a^s_{\vec{q}}\} = \{b^r_{\vec{p}}, b^s_{\vec{q}}\} = \{a^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\} = \{a^r_{\vec{p}}, b^s_{\vec{q}}\} = \ldots = 0 \,. \tag{8}$$

Show that these imply that the field and its conjugate momentum satisfy the equal time anticommutation relations

$$\begin{aligned} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} = \{\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = 0 \\ \{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y}) \,. \end{aligned} \tag{9}$$

9. Using the results of Question 8, show that the quantum Hamiltonian

$$H = \int d^3x\ \bar{\psi}(\gamma^i \partial_i + m)\psi \tag{10}$$

can be written, after normal ordering, as

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_{s=1}^{2} \left[a^{s\,\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\,\dagger}_{\vec{p}} b^s_{\vec{p}} \right] . \tag{11}$$

10. The purpose of this question is to give you a glimpse into the spin-statistics theorem. This theorem roughly says that if you try to quantize a field with the wrong statistics, bad things will happen. Here we'll see what goes wrong if you try to quantize a spin 1/2 Dirac field as a boson. We start with the usual decomposition (6). This time we choose bosonic commutation relations for the annihilation and creation operators,

$$\begin{aligned} [a^r_{\vec{p}}, a^{s\,\dagger}_{\vec{q}}] &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q}) \\ [b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}] &= -(2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q}) \end{aligned} \tag{12}$$

with all other commutators vanishing. Note the strange minus sign for the b operators. Repeat the calculation of Question 5 to show that these are equivalent to the commutation relations

$$\begin{aligned}[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = [\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = 0 \\ [\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x} - \vec{y})\,.\end{aligned} \tag{13}$$

Now repeat the calculation of Question 6, to show that, after normal ordering, the Hamitonian is given by

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}} \sum_{s=1}^{2} \left[a^{s\,\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\,\dagger}_{\vec{p}} b^s_{\vec{p}} \right] \,. \tag{14}$$

This Hamiltonian is not bounded below: you can lower the energy indefinitely by creating more and more b particles. This is the reason a theory of bosonic spin 1/2 particles is sick.

11. Using the methods presented in the lectures, find an expression for the Feynman propagator of a Dirac field

$$S_F(x-y) \equiv \langle 0|\mathrm{T}\psi(x)\overline{\psi}(y)|0\rangle \tag{15}$$

in terms of the θ-function and integrals over 3-momentum.

Deduce, by evaluating a suitable contour integral, that

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip.(x-y)} \frac{i\gamma.p + m}{p^2 + m^2 - i\epsilon} \,. \tag{16}$$

Verify that S_F is a Green's function for the Dirac operator.

12. The Lagrangian density for an electromagnetic (Maxwell) field is

$$\mathcal{L} = -\frac{1}{4} F_{ab}F^{ab} \tag{17}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ and A_a is the 4-vector potential. Show that $\mathcal{L}$ is invariant under gauge transformations

$$A_a \to A_a + \partial_a \xi \tag{18}$$

where $\xi = \xi(x)$ is a scalar field with arbitrary (differentiable) dependence on x.

Use Noether's theorem, and the spacetime translational invariance of the action, to construct the energy-momentum tensor T^{ab} for the electromagnetic field. Show that the resulting object is neither symmetric nor gauge invariant. Consider a new tensor given by

$$\Theta^{ab} = T^{ab} - F^{ca}\,\partial_c A^b\,. \tag{19}$$

Show that this object also defines four conserved currents. Moreover, show that it is symmetric, gauge invariant and traceless.

Comment: T^{ab} and Θ^{ab} are both equally good definitions of the energy-momentum tensor. However Θ^{ab} clearly has the nicer properties. Moreover, if you couple an electromagnetic field to general relativity then it is Θ^{ab} which appears in Einstein's equations.

13. The Lagrangian density for a massive vector field C_a is given by

$$\mathcal{L} = -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}m^2 C_a C^a \tag{20}$$

where $F_{ab} = \partial_a C_b - \partial_b C_a$. Derive the equations of motion and show that when $m \neq 0$ they imply

$$\partial_a C^a = 0\,. \tag{21}$$

Further show that C_0 can be eliminated completely in terms of the other fields by solving

$$\partial_i \partial^i\, C_0 + m^2 C_0 = \partial^i \dot{C}_i\,. \tag{22}$$

Construct the canonical momenta Π_i conjugate to C_i, $i = 1, 2, 3$ and show that the canonical momentum conjugate to C_0 is vanishing. Construct the Hamiltonian density $\mathcal{H}$ in terms of C_0, C_i and Π_i. (Note: Do not be concerned that the canonical momentum for C_0 is vanishing. C_0 is non-dynamical — it is determined entirely in terms of the other fields using equation (22).)