Quantum Field Theory
Example Sheet 1
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Note: in the present solutions, we are using the 'mostly plus' metric convention, i.e. $\eta = \text{diag}(-,+,+,+)$. Also, the current version has been updated to use the conserved current definition that was used class, $j^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a + F^\mu$

Exercise 1

Lagrangian:

$$L = \int_0^a dx \left[\frac{\sigma}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2\right] \tag{1}$$

Express $y(x,t)$ as a Fourier series:

$$y(x,t) = \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty} q_n(t)\sin\left(\frac{n\pi x}{a}\right). \tag{2}$$

The derivative of $y(x,t)$ with respect to x and t is

$$\frac{\partial y(x,t)}{\partial t} = \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty} \dot{q}_n(t)\sin\left(\frac{n\pi x}{a}\right) \tag{3}$$

$$\frac{\partial y(x,t)}{\partial x} = \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty} \frac{n\pi}{a} q_n(t)\cos\left(\frac{n\pi x}{a}\right), \tag{4}$$

where we abbreviate $\dot{q} \equiv \partial q/\partial t$. Substituting Eqs. (3) and (4) in (12) gives

$$L = \frac{1}{a}\sum_{n,m}\int_0^a dx \left[\sigma \dot{q}_n(t)\dot{q}_m(t)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) - T\frac{nm\pi^2}{a^2}q_n(t)q_m(t)\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\right]. \tag{5}$$

Using the orthonormality relations

$$\frac{2}{a}\int_0^a dx\, \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) = \delta_{nm} \quad \text{and}$$
$$\frac{2}{a}\int_0^a dx\, \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right) = \delta_{nm} \tag{6}$$

we see that the terms with $n \neq m$ vanish. We obtain:

$$L = \sum_n \int_0^a dx \left[\frac{\sigma}{2}\dot{q}_n^2(t) - \frac{T}{2}\left(\frac{n\pi}{a}\right)^2 q_n^2(t)\right]. \tag{7}$$

The Euler-Lagrange equations are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = 0, \quad n = 0,1,2,\ldots. \tag{8}$$

With

$$\frac{\partial L}{\partial \dot{q}_n} = \sigma \dot{q}_n \quad \text{and} \tag{9}$$

$$\frac{\partial L}{\partial q_n} = -T\left(\frac{n\pi}{a}\right)^2 q_n \tag{10}$$

we obtain the equations of motion:

$$\ddot{q}_n(t) + \frac{T}{\sigma}\left(\frac{n\pi}{a}\right)^2 q_n(t) = 0, \tag{11}$$

where $n = 0,1,2,\ldots$. These are the equations of motion of harmonic oscillators with frequency $\omega_n = \frac{n\pi}{a}\sqrt{\frac{T}{\sigma}}$.

Exercise 2

We are given the classical Hamiltonian for a string:

$$H = \sum_{n=0}^{\infty} \left[\frac{1}{2}p_n^2 + \frac{1}{2}\omega_n^2 q_n^2\right] \tag{12}$$

Upon quantization, the p_n and q_n are promoted to Hermitian operators, and we impose the canonical commutation relations upon them:

$$[\hat{q}_n,\, \hat{q}_m] \;=\; [\hat{p}_n,\, \hat{p}_m] \;=\; 0 \qquad \text{and} \qquad [\hat{q}_n,\, \hat{p}_m] = i\delta_{nm}\,.$$

We then introduce the creation and annihilation operators $\hat{a}_n^\dagger$ and $\hat{a}_n$ as a decomposition of $\hat{p}_n$ and $\hat{q}_n$:

$$\hat{q}_n = \frac{1}{\sqrt{2\omega_n}}\left(\hat{a}_n + \hat{a}_n^\dagger\right) \qquad \text{and} \qquad \hat{p}_n = i\sqrt{\frac{\omega_n}{2}}\left(\hat{a}_n^\dagger - \hat{a}_n\right) \tag{13}$$

$$\text{or (inverting these)} \quad \hat{a}_n \equiv a_n = \sqrt{\frac{\omega_n}{2}}q_n + \frac{i}{\sqrt{2\omega_n}}p_n \qquad \text{and} \qquad \hat{a}_n^\dagger \equiv a_n^\dagger = \sqrt{\frac{\omega_n}{2}}q_n - \frac{i}{\sqrt{2\omega_n}}p_n$$

Where we have dropped the hats in the last line to simplify the notation. To check that such definitions of a_n's and $a_n^\dagger$'s indeed satisfy the canonical commutation relations, we compute:

$$\begin{aligned}
[a_n,\, a_m] &= \left[\sqrt{\frac{\omega_n}{2}}q_n + \frac{i}{\sqrt{2\omega_n}}p_n,\; \sqrt{\frac{\omega_n}{2}}q_m + \frac{i}{\sqrt{2\omega_m}}p_m\right] \\
&= \frac{\sqrt{\omega_n\omega_m}}{2}[q_n,\, q_m] + \frac{i}{2}\sqrt{\frac{\omega_m}{\omega_n}}[p_n,\, q_m] + \frac{i}{2}\sqrt{\frac{\omega_n}{\omega_m}}[q_n,\, p_m] - \frac{1}{2\sqrt{\omega_n\omega_m}}[p_n,\, p_m] \\
&= +\frac{i}{2}(-i\delta_{nm}) + \frac{i}{2}i\delta_{nm} \\
&\equiv 0\,.
\end{aligned} \tag{14}$$

Taking the adjoint of this latter commutator, we find:

$$[a_n^\dagger,\, a_m^\dagger] \equiv 0\,. \tag{15}$$

The remaining commutator is

$$\begin{aligned}
[a_n,\, a_m^\dagger] &= \left[\sqrt{\frac{\omega_n}{2}}q_n + \frac{i}{\sqrt{2\omega_n}}p_n,\; \sqrt{\frac{\omega_n}{2}}q_n - \frac{i}{\sqrt{2\omega_n}}p_n\right] \\
&= \frac{-i}{2}\sqrt{\frac{\omega_n}{\omega_m}}[q_n,\, p_m] + \frac{i}{2}\sqrt{\frac{\omega_m}{\omega_n}}[p_n,\, q_m] \\
&= \frac{-i}{2}\sqrt{\frac{\omega_n}{\omega_m}}(i\delta_{mn}) + \frac{i}{2}\sqrt{\frac{\omega_m}{\omega_n}}(-i\delta_{nm}) \\
&= \delta_{mn}
\end{aligned} \tag{16}$$

Now, using (13), and plugging into the expression for the Hamiltonian, we get:

$$\begin{aligned}
H &= \sum_{n=1}^{\infty}\left[\frac{1}{2}\left(i\sqrt{\frac{\omega_n}{2}}\left(\hat{a}_n^\dagger - \hat{a}_n\right)\right)\left(i\sqrt{\frac{\omega_n}{2}}\left(\hat{a}_n^\dagger - \hat{a}_n\right)\right) + \frac{\omega_n^2}{2}\left(\frac{1}{\sqrt{2\omega_n}}\left(\hat{a}_n + \hat{a}_n^\dagger\right)\right)\left(\frac{1}{\sqrt{2\omega_n}}\left(\hat{a}_n + \hat{a}_n^\dagger\right)\right)\right] && (17) \\
&= \sum_{n=1}^{\infty}\left[-\frac{\omega_n}{4}\left(\hat{a}_n^\dagger\hat{a}_n^\dagger - \hat{a}_n\hat{a}_n^\dagger - \hat{a}_n^\dagger\hat{a}_n + \hat{a}_n\hat{a}_n\right) + \frac{\omega_n}{4}\left(\hat{a}_n\hat{a}_n + \hat{a}_n^\dagger\hat{a}_n + \hat{a}_n\hat{a}_n^\dagger + \hat{a}_n^\dagger\hat{a}_n^\dagger\right)\right] && (18) \\
&= \sum_{n=1}^{\infty}\frac{\omega_n}{2}\left[\hat{a}_n\hat{a}_n^\dagger + \hat{a}_n^\dagger\hat{a}_n\right] && (19)
\end{aligned}$$

as required.

Suppose now the existence of a ground state $|0\rangle$ such that $\hat{a}_n|0\rangle = 0$ for every n. Using the commutation relation (16), we can rewrite the Hamiltonian in normal order as follows:

$$H = \sum_{n=1}^{\infty}\frac{\omega_n}{2}\left[2\hat{a}_n\hat{a}_n^\dagger + \delta_{nn}\right] = \sum_{n=1}^{\infty}\omega_n\hat{a}_n\hat{a}_n^\dagger + \sum_{n=1}^{\infty}\frac{\omega_n}{2}\delta_{nn}\,.$$

The last, infinite term is due to the vacuum energy. Indeed, we find that the vacuum expectation value is given by:

$$\langle 0|H|0\rangle = \sum_{n=1}^{\infty}\frac{\omega_n}{2}\delta_{nn}. \tag{20}$$

Physical experiments are typically only sensitive to differences of energy[1], not absolute magnitudes. This justifies us therefore neglecting the vacuum energy. Performing this subtraction, we recover the normal ordered prescription for the Hamiltonian:

$$: H := \sum_{n=1}^{\infty} = \omega_n a_n^\dagger a_n. \tag{21}$$

From this, we calculate the energy of the states $|(n,l)\rangle = \left(a_n^\dagger\right)^l |0\rangle$ containing l particles in the n-th mode. First, we may prove by induction that

$$\left[H, \left(a_n^\dagger\right)^l\right] = l\omega_n \left(a_n^\dagger\right)^l \ , \ l \geq 1\,.$$

For $l = 1$, this is a straightforward application of the commutation relation (16) to obtain that:

$$\left[H, a_n^\dagger\right] = \sum_{m=1}^{\infty} \omega_m \left[a_m^\dagger a_m, a_n^\dagger\right] = -\sum_{m=0}^{\infty} \left(a_m^\dagger(-\delta_{nm})\right) = \omega_n a_n^\dagger. \tag{22}$$

Supposing now that (22) is valid for $l \leq k$, we find

$$\begin{aligned}\left[H, \left(a_n^\dagger\right)^{k+1}\right] &= \left[H, \left(a_n^\dagger\right)^{k}\right] a_n^\dagger + \left(a_n^\dagger\right)^{k} H a_n^\dagger - \left(a_n^\dagger\right)^{k+1} H \\ &= k\omega_n \left(a_n^\dagger\right)^{k+1} + \left(a_n^\dagger\right)^{k} \left[H, a_n^\dagger\right] \\ &= k\omega_n \left(a_n^\dagger\right)^{k+1} + \omega_n \left(a_n^\dagger\right)^{k+1} \\ &= (k+1)\omega_n \left(a_n^\dagger\right)^{k+1} \end{aligned} \tag{23}$$

as required. Having removed the vacuum energy, we now have $H|0\rangle = 0$, so the energies of the states $|(n,l)\rangle$ are:

$$H|(n,l)\rangle = \left(\left[H, \left(a_n^\dagger\right)^l\right] + \left(a_n^\dagger\right)^l H\right)|0\rangle = l\omega_n \left(a_n^\dagger\right)^l |0\rangle = l\omega_n|(n,l)\rangle \tag{24}$$

Using this, we can now move on to compute the energy of the Fock state of the form

$$|(l_1, l_2, ..., l_N)\rangle = \left(a_1^\dagger\right)^{l_1} \left(a_2^\dagger\right)^{l_2} ... \left(a_N^\dagger\right)^{l_N} |0\rangle \tag{25}$$

Acting on these with H to obtain:

$$\begin{aligned} H|(l_1, l_2, ..., l_N)\rangle &= \left(\left[H, \left(a_1^\dagger\right)^{l_1}\right] + \left(a_1^\dagger\right)^{l_1} H\right)|(l_1, l_2, ..., l_N)\rangle \\ &= l_1\omega_1|l_1, l_2, ..., l_n\rangle + \left(a_1^\dagger\right)^{l_1}\left(H|(l_1, l_2, ..., l_N)\rangle\right) \end{aligned} \tag{26}$$

This relation can be applied iteratively, to find

$$H|(l_1, l_2, ..., l_N)\rangle = \left(\sum_{i=1}^{N} l_i\omega_i\right)|(l_1, l_2, ..., l_N)\rangle + \left(\prod_{i=1}^{N}\left(a_i^\dagger\right)^{l_i}\right) H|0\rangle = \left(\sum_{i=1}^{N} l_i\omega_i\right)|(l_1, l_2, ..., l_N)\rangle \tag{27}$$

i.e. the energy of the multi-particle state $|(l_1, l_2, ..., l_N)\rangle$ is $\sum_{i=1}^N l_i\omega_i$.

Exercise 3

The scalar field $\phi(x)$ transforms under a Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ as

$$\phi(x) \to \phi'(x) = \phi(y) = \phi(\Lambda^{-1}x).$$

Note that this is an active transformation, i.e. the fields are transformed but the coordinates are left unchanged, and the new field equals the old field at the coordinates transformed backward. Using

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu} = (\Lambda^{-1})^\nu{}_\mu \partial'_\nu$$

Now,

$$\begin{aligned} -\partial^\mu\partial_\mu\phi(x) + m^2\phi(x) \quad \to \quad & -\partial^\mu\partial_\mu\phi'(x) + m^2\phi'(x) \\ &= -\eta^{\mu\nu}(\Lambda^{-1})^\rho{}_\mu\partial'_\rho(\Lambda^{-1})^\kappa{}_\nu\partial'_\kappa\phi(y) + m^2\phi(y) \\ &= -\eta^{\rho\kappa}\partial'_\rho\partial'_\kappa\phi(x) + m^2\phi(y). \end{aligned} \tag{28}$$

[1] Note that this does not hold in systems where gravitational interactions are considered, since in this case the energy-momentum tensor - enclosing the total energy density - sources the Einstein equations

In the last step we used that Λ is a Lorentz transformation and so its inverse Λ^{-1} preserves the inverse Minkowski metric $\eta^{\nu\mu}$,

$$\left(\Lambda^{-1}\right)^{\rho}{}_{\mu}\eta^{\mu\nu}\left(\Lambda^{-1}\right)^{\kappa}{}_{\nu} = \eta^{\rho\kappa} .$$

Renaming y to be x in Eq. (28), this equation demonstrates that if $\phi(x)$ fulfils the Klein-Gordon equation

$$-\partial^{\mu}\partial_{\mu}\phi(x) + m^2\phi(x) = 0 ,$$

then $\phi(\Lambda^{-1}x)$ fulfils it as well.

Exercise 4

Lagrangian density:

$$\mathcal{L} = -\partial_{\mu}\phi^*\partial^{\mu}\phi - m^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2. \tag{29}$$

Euler-Lagrange equation for ϕ^*:

$$\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^*)} = 0. \tag{30}$$

With

$$\frac{\partial\mathcal{L}}{\partial\phi^*} = -m^2\phi - \lambda(\phi^*\phi)\phi \quad \text{and} \tag{31}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^*)} = -\partial^{\mu}\phi \tag{32}$$

we obtain the equation of motion for ϕ^*:

$$-\partial_{\mu}\partial^{\mu}\phi + m^2\phi + \lambda(\phi^*\phi)\phi = 0. \tag{33}$$

Similarly, we can calculate the field equation for ϕ. The result is the complex conjugate of Eq. (33).
Consider the U(1) transformation

$$\phi(x) \to e^{i\alpha}\phi(x), \quad \text{and} \quad \phi^{\dagger}(x) \to e^{i\alpha}\phi^*(x), \tag{34}$$

where $\alpha \in [0, 2\pi)$ is a constant. It is straightforward to check that the Lagrangian (29) is invariant under this transformation. This is an example of a *global* symmetry, i.e., the symmetry transformation acts in the same way on the fields at each point in space and time.
The infinitesimal transformation of (34) is:

$$\phi \to \phi + \delta\phi, \quad \text{where} \quad \delta\phi = i\alpha\phi, \quad \text{and} \tag{35}$$

$$\phi^* \to \phi^* + \delta\phi^*, \quad \delta\phi^* = -i\alpha\phi. \tag{36}$$

We can check explicitly that the Lagrangian density is invariant under this transformation:

$$\begin{aligned}\delta\mathcal{L} &= \partial_{\mu}(\delta\phi^*)\partial^{\mu}\phi + \partial_{\mu}\phi^*\partial^{\mu}(\delta\phi) - m^2(\delta\phi^*\phi + \phi^*\delta\phi) - \lambda(\phi^*\phi)(\phi^*\delta\phi + \delta\phi^*\phi) \\ &= -i\alpha\partial_{\mu}\phi^*\partial^{\mu}\phi + i\alpha\partial_{\mu}\phi^*\partial^{\mu}\phi + i\alpha m^2(\phi^*\phi - \phi^*\phi) - i\alpha\lambda(\phi^*\phi)(\phi^*\phi - \phi^*\phi) = 0.\end{aligned} \tag{37}$$

According to Noether's theorem, the global symmetry implies the existence of a conserved current. If $\mathcal{L} \to \mathcal{L} + \alpha\Delta\mathcal{L}$, where $\Delta\mathcal{L} = \partial_{\mu}\mathcal{J}^{\mu}$, then the conserved current is given by the formula:

$$j^{\mu} = -\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_a(x))}\Delta\phi_a + \mathcal{J}^{\mu} \tag{38}$$

where a labels all the fields in the theory. In the present case, there are two independent fields ϕ and ϕ^*, and

$$\Delta\phi = i\phi\,,\ \Delta\phi^* = -i\phi\,,\ \Delta\mathcal{L} = 0\,. \tag{39}$$

The concerned current is thus:

$$\begin{aligned}j^{\mu} &= -\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}(i\phi) - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^*)}(-i\phi^*) \\ &= -i\left(\phi^*(\partial^{\mu}\phi) - (\partial^{\mu}\phi^*)\phi\right).\end{aligned} \tag{40}$$

We can check explicitly that this current is conserved:

$$\begin{aligned}\partial_{\mu}j^{\mu} &= -i\left(\phi^*\partial_{\mu}(\partial^{\mu}\phi) - (\partial^{\mu}\phi^*)(\partial_{\mu}\phi) + (\partial_{\mu}\phi^*)(\partial^{\mu}\phi) - \partial_{\mu}(\partial^{\mu}\phi^*)\phi\right) \\ &= -i\left(\phi^*\partial_{\mu}(\partial^{\mu}\phi) - \partial_{\mu}(\partial^{\mu}\phi^*)\phi\right) \\ &= -i\left(\phi^*\left[m^2\phi + \lambda(\phi^*\phi)\phi\right] - \left[m^2\phi^* + \lambda(\phi^*\phi)\phi^*\right]\phi\right) = 0.\end{aligned} \tag{41}$$

In the last step, we used the equation of motion for ϕ and ϕ^*, Eq. (33).

Exercise 5

Intuitively we expect the Lagrangian to be invariant under SO(3) rotations of the fields ϕ_a, $a = 1, 2, 3$, since only the 'length' $\phi_a\phi_a$ enters the Lagrangian. To see this explicitly, we consider the transformation of $\phi_a\phi_a$ under a rotation of the fields ϕ_a by an infinitesimal angle θ

$$\phi_a \to \phi_a + \theta\epsilon_{abc}n_b\phi_c\,, \qquad i.e.\;\; \Delta\phi_a = \epsilon_{abc}n_b\phi_c \tag{42}$$

where n_a is a constant unit vector. $\phi_a\phi_a$ transforms as

$$\begin{aligned}\phi_a\phi_a \to \quad & (\phi_a + \theta\epsilon_{abc}n_b\phi_c)(\phi_a + \theta\epsilon_{ade}n_d\phi_e)\\ &= \phi_a\phi_a + \theta\epsilon_{abc}n_b\phi_c\phi_a + \theta\epsilon_{ade}n_d\phi_e\phi_a + \mathcal{O}(\theta^2)\\ &= \phi_a\phi_a + \mathcal{O}(\theta^2).\end{aligned}$$

In the last step we used that ϵ_{abc} is skew-symmetric under exchange of c and a whereas $\phi_c\phi_a$ is symmetric under this exchange, and therefore the sum vanishes. The same applies to the other term linear in θ. Dropping the θ^2 term we find that $\phi_a\phi_a$ is indeed invariant under (42). Similarly one can also show that $\partial_\mu\phi_a\partial^\mu\phi_a$ is invariant under (42). Therefore $\mathcal{L}$ is invariant under (42).

To obtain the associated Noether current, we refer to the formula (38) above. In our case $\mathcal{J} = 0$ and $\Delta\phi_a = \epsilon_{abc}n_b\phi_c$, so that

$$j^\mu = -(\partial^\mu\phi_a)\epsilon_{abc}n_b\phi_c.$$

The Noether current j^μ implies a conserved charge

$$Q \equiv \int d^3x\, j^0 = -\int d^3x\, \dot{\phi}_a\epsilon_{abc}n_b\phi_c.$$

This is conserved ($\dot{Q} = 0$) for any unit vector n. If we choose $n_b = \delta_{bd}$ for $d = 1, 2, 3$ (i.e. n is any of the standard basis vector of $\mathbb{R}^3$), we obtain three linearly independent charges

$$Q_d = -\int d^3x\, \epsilon_{adc}\dot{\phi}_a\phi_c = \int d^3x\, \epsilon_{dac}\dot{\phi}_a\phi_c\,.$$

Let us finally show explicitly that these charges are conserved using the equations of motion. These are given by

$$-\partial_\mu\partial^\mu\phi_a + m^2\phi_a = \ddot{\phi}_a - \nabla^2\phi_a + m^2\phi_a = 0.$$

Then we get

$$\begin{aligned}\frac{\mathrm{d}Q_a}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int d^3x\, \epsilon_{abc}\dot{\phi}_b\phi_c\\ &= \int d^3x\, \epsilon_{abc}(\ddot{\phi}_b\phi_c + \dot{\phi}_b\dot{\phi}_c)\\ &= \int d^3x\, \epsilon_{abc}(\ddot{\phi}_b\phi_c)\\ &= \int d^3x\, \epsilon_{abc}\left[(\nabla^2\phi_b)\,\phi_c - m^2\phi_b\phi_c\right]\\ &= -\int d^3x\, \epsilon_{abc}\nabla_i\phi_b\nabla^i\phi_c = 0.\end{aligned}$$

Here we used the antisymmetry of the ϵ tensor several times and integrated by parts to go to the last line. We assume that the field falls off sufficiently fast so that we can neglect the boundary term of the partial integration.

Exercise 6

Under a Lorentz transformation, the vector x^μ transforms as:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu\,, \tag{43}$$

and such transformations preserve the metric, i.e. we know

$$g_{\mu\nu}x^\mu x^\nu = g_{\mu\nu}x'^\mu x'^\nu\,. \tag{44}$$

But replacing the x'^μs by their definition in terms of $\Lambda^\mu{}_\nu$,

$$g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\left(\Lambda^\mu{}_\alpha x^\alpha\right)\left(\Lambda^\nu{}_\beta x^\beta\right) = \left(g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\right)x^\alpha x^\beta = \left(g_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu\right)x^\mu x^\nu\,. \tag{45}$$

In the last step, we have interchanged the indices α and β with μ and ν. This is allowed since they are contracted, and therefore this is just a renaming. Now, equating (44) with the last step in (45), it follows straightforwardly that

$$g_{\mu\nu} = g_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\,. \tag{46}$$

Now, let us consider an infinitesimal Lorentz transformation of the form

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}\,, \qquad |\omega| \ll 1\,. \tag{47}$$

Using (46), we find that, keeping only the terms up to first order in ω

$$g_{\mu\nu} = \left(\delta^{\alpha}{}_{\mu} + \omega^{\alpha}{}_{\mu}\right) g_{\alpha\beta}\left(\delta^{\beta}{}_{\nu} + \omega^{\beta}{}_{\nu}\right) = g_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} + \mathcal{O}\left(\omega^2\right)\,, \tag{48}$$

which requires that $\omega_{\mu\nu}$ is an antisymmetric tensor for the above transformation to indeed be a Lorentz transformation.

A tangent

For those of you who are interested, we introduce an alternative, perhaps more mathematically sound means of thinking about the concept of an 'infinitesimal' Lorentz transformation. Consider a one-parameter family of Lorentz transformations, in particular a differentiable map $\Lambda : \mathbb{R} \to \mathrm{SO}(3,1)$, i.e. for each value of the parameter t, $\Lambda(t)$ defines a Lorentz transformation. Now define the parameter t such that $\Lambda(0) = I$, the identity transformation.
Then, for all t,

$$g_{\mu\nu} = g_{\alpha\beta}\Lambda(t)^{\alpha}{}_{\mu}\Lambda(t)^{\beta}{}_{\nu}\,. \tag{49}$$

Looking 'infinitesimally' corresponds to differentiating at the identity, i.e. consider

$$\begin{aligned}\frac{d}{dt}\left(g_{\alpha\beta}\Lambda(t)^{\alpha}{}_{\mu}\Lambda(t)^{\beta}{}_{\nu}\right)\Big|_{t=0} &= g_{\alpha\beta}\Lambda'(0)^{\alpha}{}_{\mu}\Lambda(0)^{\beta}{}_{\nu} + g_{\alpha\beta}\Lambda(0)^{\alpha}{}_{\mu}\Lambda'(0)^{\beta}{}_{\nu}\\ &= g_{\alpha\nu}\Lambda'(0)^{\alpha}{}_{\mu} + g_{\mu\beta}\Lambda'(0)^{\beta}{}_{\nu}\,.\end{aligned}$$

However, according to (49), this is precisely

$$\frac{d}{dt}\left(g_{\alpha\beta}\Lambda(t)^{\alpha}{}_{\mu}\Lambda(t)^{\beta}{}_{\nu}\right)\Big|_{t=0} = \frac{d}{dt}g_{\mu\nu}\Big|_{t=0} = 0\,.$$

Therefore, if we define a matrix ω, by

$$\omega^{\alpha}{}_{\mu} = \Lambda'(0)^{\alpha}{}_{\mu}\,,$$

we have in all that

$$0 = g_{\alpha\nu}\omega^{\alpha}{}_{\mu} + g_{\mu\beta}w^{\beta}{}_{\nu} = \omega_{\nu\mu} + \omega_{\mu\nu}\,.$$

To make the connection with our earlier 'infinitesimal' form of the Lorentz transformation $\Lambda^{\mu}{}_{\nu}$, we can write down the Taylor expansion of $\Lambda(t)$ for t small,

$$\begin{aligned}\Lambda(t)^{\mu}{}_{\nu} &= \Lambda(0)^{\mu}{}_{\nu} + t\,\Lambda'(0)^{\mu}{}_{\nu} + \mathcal{O}(t^2)\\ &= \delta^{\mu}{}_{\nu} + t\,\omega^{\mu}{}_{\nu} + \mathcal{O}(t^2)\,.\end{aligned}$$

Besides being rigorous, the benefit of this approach is that Λ infinitesimal is characterised by a parameter t being small, while $\omega_{\mu\nu}$ is any skew-symmetric matrix and not an infinitesimal object. One-parameter families of transformations, and in particular one-parameter subgroups (i.e. smooth *homomorphisms* $\Lambda : \mathbb{R} \to G$), are precisely how one defines the Lie algebra of a group and its exponential map in a more general setting - i.e. when the group isn't given as being embedded in $\mathrm{GL}(n, \mathbb{R})$.

The form of the infinitesimal rotations and boosts are given by:

- Rotation:
  A generic rotation by and angle θ about the x^3-axis is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{50}$$

  Taylor expanding the trigonometric functions up to first order in θ, we find that an infinitesimal rotation around the x^3-axis is given by:

$$\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & \theta & 0\\ 0 & -\theta & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{51}$$

As a check on the above, exponentiate the matrix

$$\omega = \theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

In order to do this, note that

$$\omega^2 = \theta^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

so that

$$\omega^{2k} = (-1)^k \theta^{2k} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad k \geq 1 \quad \text{and} \quad \omega^{2k+1} = (-1)^k \theta^{2k+1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad k \geq 0 .$$

Now the exponentiation can be calculated from

$$\begin{aligned} \exp(\omega) &= I + \sum_{k=1}^{\infty} \frac{1}{2k!} \omega^{2k} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \omega^{2k+1} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \left\{ \sum_{k=1}^{\infty} \frac{1}{2k!} (-1)^k \theta^{2k} \right\} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\quad + \left\{ \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} (-1)^k \theta^{2k+1} \right\} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \cos\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \end{aligned}$$

- Boost:
  A generic boost along the x^1-axis is given by:

$$\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{52}$$

Recall that the *rapidity* ϕ is defined via $\tanh\phi = v$, so that the above boosts are parameterised in analogy to the rotations by

$$\begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{53}$$

Taylor expanding to first order about $\phi = 0$ (i.e. $v = 0$), the infinitesimal boost by v is given by:

$$\delta^\mu{}_\nu + \omega^\mu{}_\nu = \delta^\mu{}_\nu + \phi \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{54}$$

We will again check that the above 'infinitesimal' transformation gives the correct finite transformation when we exponentiate the matrix

$$\omega = \phi \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Calculate the square and notice the pattern emerging

$$\omega^2 = \phi^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

so that the even and odd powers are

$$\omega^{2k} = \phi^{2k} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \; k \geq 1 \quad \omega^{2k+1} = \phi^{2k+1} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , k \geq 0 .$$

Now the exponentiation can be calculated from

$$\begin{aligned}
\exp(\omega) &= I + \sum_{k=1}^{\infty} \frac{1}{2k!} \omega^{2k} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \omega^{2k+1} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \left\{ \sum_{k=1}^{\infty} \frac{1}{2k!} \phi^{2k} \right\} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad + \left\{ \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \phi^{2k+1} \right\} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \cosh\phi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \sinh\phi \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .
\end{aligned}$$

Food for thought: There is some subtlety involved in the above operations:

1. If one takes a group, is it true that exponentiation (a representation of) its algebra gives an element of (a representation of) the original group? Seen from a different perspective, can two groups have the same algebra?
2. If the answer to the previous question is positive, can one obtain all elements in this way?

Unfortunately, the answer to both of these questions is negative. Fortunately, the answer is almost as nice as one would wish, and is certainly much more interesting than a bland affirmative.

Aspects of the relevant theory may be covered in the course 'Symemtries and Particles' and is covered in the course 'Lie Algebras and Their Representations' though, needless to say, in great abstraction. Otherwise, see Warner, *Foundations of Differentiable Manifolds* for general background of Fulton & Harris, *Representation Theory* for full details.

Exercise 7

From the above derivation, under an infinitesimal Lorentz transformation, a vector x^μ transforms as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu = x^\mu + \omega^\mu{}_\nu x^\nu . \tag{55}$$

Similarly, a scalar field ϕ transforms according to

$$\phi(x^\mu) \to \phi'(x^\mu) = \phi\left((\Lambda^{-1})^\mu{}_\nu x^\nu\right) = \phi(x^\mu - \omega^\mu{}_\nu x^\nu) . \tag{56}$$

Where in the last step we needed to know what Λ^{-1} looks like infinitesimally. To figure this out, we look at the definition that it preserves the metric

$$g_{\mu\nu} = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}g_{\alpha\beta}\,,$$

and we find that the inverse Lorentz transformation is simply given by

$$\left(\Lambda^{-1}\right)^{\mu}{}_{\nu} = g^{\mu\alpha}g_{\nu\beta}\Lambda^{\beta}{}_{\alpha}\,.$$

Note that if the metric were Euclidean, this would simply be the statement $\Lambda^T = \Lambda^{-1}$, i.e. Λ is an orthogonal matrix. We now find the inverse of the infinitesimal transformation (neglecting terms of $\mathcal{O}(\omega^2)$)

$$\begin{aligned}\left(\Lambda^{-1}\right)^{\mu}{}_{\nu} &= \delta^{\mu}{}_{\nu} + \omega^{\beta}{}_{\alpha}g^{\mu\alpha}g_{\nu\beta}\\ &= \delta^{\mu}{}_{\nu} + \omega_{\nu\alpha}g^{mu\alpha}\\ &= \delta^{\mu}{}_{\nu} - \omega_{\alpha\nu}g^{\mu\alpha}\\ &= \delta^{\mu}{}_{\nu} - \omega^{\mu}{}_{\nu}\,.\end{aligned}$$

In the second line we use the skew-symmetry of $\omega_{\nu\alpha}$.

Now, Taylor expanding ϕ to first order in $\omega_{\mu\nu}$ we obtain

$$\phi(x^\mu) \to \phi'(x^\mu) \approx \phi(x^\mu) - \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\phi(x^\mu)\,. \tag{57}$$

The Lagrangian density, $\mathcal{L}$, is a Lorentz scalar and will therefore transform in the way just found under an infinitesimal Lorentz transformation

$$\mathcal{L}(x^\mu) \to \mathcal{L}'(x^\mu) = \mathcal{L}(x^\mu) - t\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L}(x^\mu)\,, \tag{58}$$

so that $\delta\mathcal{L} = -\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L}$. If you are unconvinced by this reasoning, check explicitly that the Lagrangian for the scalar field theory, given the transformation for the scalar field ϕ, does indeed change in this way.
Now look at

$$\begin{aligned}\partial_\mu\left(\omega^{\mu}{}_{\nu}x^{\nu}\mathcal{L}\right) &= \partial_\mu\left(\omega^{\mu}{}_{\nu}x^{\nu}\right)\mathcal{L} + \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L}\\ &= \omega^{\mu}{}_{\nu}\partial_{\mu}\left(x^{\nu}\right)\mathcal{L} + \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L}\\ &= \omega^{\mu}{}_{\nu}\delta_{\mu}{}^{\nu}\mathcal{L} + \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L}\\ &= \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L}\,,\end{aligned} \tag{59}$$

where in the second step we have used that $\omega^{\mu}{}_{\nu}$ is a constant tensor, and in the last step we made use of the skew-symmetry of $\omega_{\mu\nu}$ to infer that $\omega^{\mu}{}_{\mu} = 0$. Therefore, we obtain that the variation of the Lagrangian density id a total derivative

$$\delta\mathcal{L} = -\partial_\mu\left(\omega^{\mu}{}_{\nu}x^{\nu}\mathcal{L}\right)\,. \tag{60}$$

From Noether's theorem, there must be a conserved current associated with this symmetry, given by

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi + F^\mu\,, \tag{61}$$

with F defined via $\delta\mathcal{L} = \partial_\mu F^\mu$. Inserting the expressions derived above for $\delta\phi \equiv \phi(x^\mu) - \phi'(x'^\mu)$ and F^μ, we obtain:

$$\begin{aligned}j^\mu &= -\left(\omega^{\alpha}{}_{\beta}x^{\beta}\right)\left[-\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\alpha\phi + \delta^{\mu}{}_{\alpha}\mathcal{L}\right]\\ &= -\omega^{\alpha}{}_{\beta}x^{\beta}T^{\mu}{}_{\alpha}.\end{aligned} \tag{62}$$

Here, T refers to the energy-momentum tensor. The total charge Q is given by

$$Q \equiv \int d^3x(j^0) \tag{63}$$

$$= -\int d^3x\ \omega^{\rho}{}_{\nu}x^{\nu}T^{0}{}_{\rho}\,. \tag{64}$$

- For pure spatial rotation we have $\omega^{i}{}_{j} \neq 0$ and $\omega^{0}{}_{i} = \omega^{i}{}_{0} = 0$ where $i, j = 1, \ldots 3$ are spatial indices. In this case

$$\begin{aligned}Q &= -\int d^3x\ \omega^{j}{}_{k}x^{k}T^{0}{}_{j}\\ &= -\int d^3x\ \omega_{jk}x^{k}T^{0j}\\ &= \frac{1}{2}\int d^3x\ \omega_{jk}\left(x^{j}T^{0k} - x^{k}T^{0j}\right)\\ &= \frac{\omega_{jk}}{2}\int d^3x\ \left(x^{j}T^{0k} - x^{k}T^{0j}\right)\,,\end{aligned} \tag{65}$$

where in the third line the skew-symmetry was used to discard the symmetric part of $x^k T^{0j}$.
ω_{jk} is just a constant times a parameterisation of our choice of rotation and so we will aim to remove it from the expression of the charge. Begin by realising that by its skew-symmetry one can write

$$\omega_{jk} = \epsilon_{jki} v^i \, ,$$

for some vector v^i which is normal to the rotation. This can be thought of as defining an angular momentum vector as in classical dynamics. Then

$$\begin{aligned} Q &= \frac{1}{2} v^i \epsilon_{ijk} \int d^3x \, \left(x^j T^{0k} - x^k T^{0j} \right) \\ &\equiv \frac{1}{2} Q_i v^i \, , \end{aligned}$$

and we can identify Q_i as the three linearly independent charges

$$Q_i = \epsilon_{ijk} \int d^3x \, \left(x^j T^{0k} - x^k T^{0j} \right) \, , \tag{66}$$

which have the interpretation of angular momenta.

- For a Lorentz boost we have only $\omega^0{}_i \neq 0$ and $\omega^i{}_j = 0$. A simple calculation also shows that

$$w^k{}_0 = \omega_{k0} = -\omega_{0k} = \omega^0{}_k \, .$$

Therefore in this case

$$\begin{aligned} Q &= -\int d^3x \, \left(\omega^0{}_i x^i T^0{}_0 + \omega^i{}_0 x^0 T^0{}_i \right) \\ &= -\omega^0{}_i \int d^3x \, \left(x^i T^0{}_0 - x^0 T^i{}_0 \right) \\ &= \omega_{0i} \int d^3x \, \left(x^0 T^{0i} - x^i T^{00} \right) \, , \end{aligned}$$

where for the 0 raised and that lowered we each get a -1 so there is no overall sign change. Since there is such a conserved quantity for each choice of ω_{0i} and there are three such independent choices we find the 3-vector of conserved charges

$$Q^i = \int d^3x \, \left(x^0 T^{0i} - x^i T^{00} \right) \, .$$

Since Q^i is conserved, taking its derivative with respect to time gives zero

$$\begin{aligned} \frac{d}{dt} Q^i = 0 \Rightarrow 0 &= \frac{d}{dt} \left(\int d^3x \, x^0 T^{i0} \right) - \frac{d}{dt} \left(\int d^3x \, x^i T^{00} \right) \\ &= \int d^3x \, T^{i0} + t \int d^3x \, \frac{d}{dt} \left(T^{i0} \right) - \frac{d}{dt} \left(\int d^3x \, x^i T^{00} \right) \, , \\ \Rightarrow \frac{d}{dt} \left(\int d^3x \, x^i T^{00} \right) &= \int d^3x \, T^{i0} + t \int d^3x \, \frac{d}{dt} \left(T^{i0} \right) \, . \end{aligned}$$

Let us investigate the terms in this equation separately. T^{00} is the energy density of the quanum field while T^{0k} is its linear momentum density. Because the theory is invariant under translations, the total linear momentum must be conserved thus $\int d^3x \, T^{i0} = const.$ Similarly, this means that the second term on the right-hand side vanishes. So we simply get

$$\frac{d}{dt} \left(\int d^3x \, x^i T^{00} \right) = \int d^3x \, T^{i0} = const.$$

This has a simple interpretation: the integral on the LHS gives the centre of energy of the field and so the equation is telling us that it moves at a constant velocity.

Exercise 8

Let us begin in the greatest generality, i.e. work in (n+1)-dimensional Minkowski space-time. We are given that the group $(\mathbb{R}_+, \times)$ of positive reals under multiplication acts on the space of field configurations via the action

$$(\lambda \cdot \phi)(x) \equiv \lambda^{-D} \phi(\lambda^{-1} x)$$

for some D, the *scaling dimension* of ϕ. In other (perhaps less technical) words, we have a transformation

$$\phi(x) \longmapsto \lambda^{-D}\phi(\lambda^{-1}x) \ , \ \lambda > 0.$$

We show that this is a symmetry of the action

$$S[\phi] = \int d^{n+1}x \, \mathcal{L}\left(\phi(x), \partial\, \phi(x)\right),$$

where the Lagrangian density is given as

$$\mathcal{L}\left(\phi(x), \partial\, \phi(x)\right) = -\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi(x)^2 - g\phi(x)^p,$$

for constants m, p and g. We assume $p \neq 2$, else the third term simply serves to modify the mass.

Recall that a group action is a symmetry iff we have

$$S[\lambda \cdot \phi] = S[\phi] \ , \ \forall \lambda \in \mathbb{R}_+,$$

i.e. if the above transformation leaves S unchanged. So we must show that

$$S[\lambda\cdot\phi] = \int d^{n+1}x \left\{ -\frac{1}{2}\lambda^{-2D}\partial_\mu\left[\phi(\lambda^{-1}x)\right]\partial^\mu\left[\phi(\lambda^{-1}x)\right] - \frac{1}{2}m^2\lambda^{-2D}\phi(\lambda^{-1}x)^2 - g\lambda^{-pD}\phi(\lambda^{-1}x)\right\}$$

does not depend on λ. Making the subtitution $x' = \lambda^{-1}x$, then as $\lambda > 0$,

$$d^{n+1}x = |\lambda^{n+1}|\, d^{n+1}x' = \lambda^{n+1}d^{n+1}x' \ , \ \partial_\mu = \lambda^{-1}\frac{\partial}{\partial x'^\mu}$$

and, abbreviating

$$\frac{\partial}{\partial x'^\mu} = \partial'_\mu,$$

we find

$$S[\lambda\cdot\phi] = \int d^{n+1}x' \, \lambda^{n+1}\left\{-\frac{1}{2}\lambda^{-2D-2}\partial'_\mu\phi(x')\partial'^\mu\phi(x') - \frac{1}{2}m^2\lambda^{-2D}\phi(x')^2 - g\lambda^{-pD}\phi(x')\right\},$$

which, since we can replace x' with x as a dummy variable in the integral, is the original value of the action iff each term in the integrand,

$$\lambda^{-2D-2+n+1}\partial_\mu\phi\,\partial^\mu\phi \ , \ m^2\lambda^{-2D+n+1}\phi^2 \ , \ g\lambda^{-pD+n+1}\phi^p,$$

does not depend on λ. Invariance of the first term entails

$$D = \frac{1}{2}(n-1).$$

Having found D, invariance of the second term occurs iff $m = 0$. Finally, the third term is invariant either when $g = 0$, else if $g \neq 0$, then

$$p = 2\frac{n+1}{n-1}.$$

In the case $n = 3$, therefore $D = 1$ and $p = 4$.

Having established that this is a symmetry, we find the conserved current associated to the 'infinitesimal' transformation in the case $n = 3$, i.e. we look at the first order changes in the fields and Lagrangian. The change in ϕ to first order is, by the chain rule,

$$\begin{aligned} \Delta\phi(x) &= \frac{d}{d\lambda}(\lambda\cdot\phi)(x)\Big|_{\lambda=1} \\ &= \frac{d}{d\lambda}\left(\lambda^{-1}\phi(\lambda^{-1}x)\right)\Big|_{\lambda=1} \\ &= -\phi(x) - x^\sigma\partial_\sigma\phi(x). \end{aligned}$$

The Lagrangian transforms as

$$\mathcal{L}\left((\lambda\cdot\phi)(x), \left[\partial(\lambda\cdot\phi)\right](x)\right) = \lambda^{-4}\mathcal{L}\left(\phi(\lambda^{-1}x), \partial\phi\left(\lambda^{-1}x\right)\right)$$

Since $\mathcal{L}$ on the right depends on λ only through the space-time dependence of the fields, we find the 'infinitesimal change' in $\mathcal{L}$ to be, again by the chain rule,

$$\begin{aligned}\Delta\mathcal{L} &= \frac{d}{d\lambda}\left\{\mathcal{L}\left((\lambda\cdot\phi)(x),\left[\partial(\lambda\cdot\phi)\right](x)\right)\right\}\Big|_{\lambda=1} \\ &= -4\mathcal{L}\left(\phi(x),\partial\phi\,(x)\right) - x^{\sigma}\partial_{\sigma}\left[\mathcal{L}\left(\phi(x),\partial\phi\,(x)\right)\right].\end{aligned}$$

As was to be expected from general principles, this is precisely the divergence

$$\Delta\mathcal{L} = \partial_{\sigma}\mathcal{J}^{\sigma}\ ,\ \text{where } \mathcal{J}^{\sigma} = -x^{\sigma}\mathcal{L}.$$

The conserved current provided by Noether's theorem is,

$$j^{\mu} = -\frac{\partial\mathcal{L}}{\partial\,\partial_{\mu}\phi}\Delta\phi + \mathcal{J}^{\mu}.$$

Plugging in the expressions found above, we eventually obtain the conserved current associated to our scaling symmetry

$$j^{\mu} = -\partial^{\mu}\phi\left(\phi + x^{\sigma}\partial_{\sigma}\phi\right) - x^{\mu}\mathcal{L}.$$

Once the field equations have been found, it is straightforward to check that this is indeed conserved.

Exercise 9

In the Heisenberg picture the fields are time-dependent. This is related to the Schrodinger picture by

$$\phi_H(t,\vec{x}) = e^{iHt}\phi_S(\vec{x})e^{-iHt}$$

We omit the subscripts and use the four-vector notation $\phi(x)$ to distinguish the between the two pictures. The Hamiltonian for the system is given by

$$H = \frac{1}{2}\int dy^3\left(\pi^2(y) + |\nabla\phi(y)|^2 + m^2\phi^2(y)\right)$$

This system is quantised by imposing equal time commutation relations on $\phi(y)$ and $\pi(y)$ in the Heisenberg picture as,

$$[\phi(t,\vec{x}),\phi(t,\vec{y})] = [\pi(t,\vec{x}),\pi(t,\vec{y})] = 0\ \ ,\ \ [\phi(t,\vec{x}),\pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$$

Now it follows that

$$\dot{\phi}(x) = iH(e^{iHt}\phi(\vec{x})e^{-iHt}) - i(e^{iHt}\phi(\vec{x})e^{-iHt})H = i[H,\phi(x)]$$

In terms of the mode decompositions, the only non-vanishing piece is the commutator of π and ϕ.

$$i[H,\phi(x)] = i\left[\frac{1}{2}\int dy^3(\pi^2(y) + |\nabla\phi(y)|^2 + m^2\phi^2(y)),\phi(x)\right] = \frac{i}{2}\int dy^3[\pi^2(y),\phi(x)] = \pi(x) \tag{67}$$

where we have used the equal time commutation relations and the fact that in $\nabla\phi(y)$ the derivative acts on $\vec{y}$ and hence can be pulled out of the commutator $[\nabla\phi(y),\phi(x)]$ which then equals zero. Similarly, for the other case we have,

$$\begin{aligned}\dot{\pi}(x) = i[H,\pi(x)] &= \frac{i}{2}\int dy^3\left[\pi^2(y) + |\nabla\phi(y)|^2 + m^2\phi^2(y),\pi(x)\right] \\ &= i^2\int dy^3\left[\nabla\phi(y).\nabla\delta^{(3)}(\vec{y}-\vec{x}) + m^2\phi(y)\delta^{(3)}(\vec{y}-\vec{x})\right] \\ &= \nabla^2\phi(x) - m^2\phi(x)\end{aligned} \tag{68}$$

It now follows from Eq.(67) and Eq.(68) that $\phi(x)$ satisfies the Klein-Gordon equation $-\ddot{\phi}(x) + \nabla^2\phi(x) = m^2\phi^2(x)$

Exercise 10

The relativistically normalised one particle states are $|p\rangle = \sqrt{2E_{\vec{p}}}\,a^{\dagger}_{\vec{p}}|0\rangle$. Using

$$\phi(x) = \int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{q}}}}(a_{\vec{q}}e^{iqx} + a^{\dagger}_{\vec{q}}e^{-iqx})$$

we find

$$\begin{aligned}
\langle 0|\phi(x)|p\rangle &= \int \frac{d^3q}{(2\pi)^3}\sqrt{\frac{2E_{\vec{p}}}{2E_{\vec{q}}}}\langle 0|(a_{\vec{q}}e^{iqx} + a^\dagger_{\vec{q}}e^{-iqx})a^\dagger_{\vec{p}}|0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\sqrt{\frac{2E_{\vec{p}}}{2E_{\vec{q}}}}e^{iqx}\langle 0|a_{\vec{q}}a^\dagger_{\vec{p}}|0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3}\sqrt{\frac{2E_{\vec{p}}}{2E_{\vec{q}}}}e^{iqx}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\langle 0|0\rangle \\
&= e^{ipx}
\end{aligned}$$

Where we have used $\langle 0|a^\dagger_{\vec{q}} = 0$, $[a_{\vec{q}}, a^\dagger_{\vec{p}}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$ and $\langle 0|0\rangle = 1$. Comparing this with the relation $\langle \vec{x}|\vec{p}\rangle = e^{i\vec{p}.\vec{x}}$ from quantum mechanics, we interpret $\phi(x)|0\rangle$ as a particle at x.