Quantum Field Theory
Example Sheet 3
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Solutions by:

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Exercise 1

Let us start from the Maxwell action:

$$\int d^4x \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \,. \tag{1}$$

The equations of motion are then:

$$\partial_\mu F^{\mu\nu} = 0 \,. \tag{2}$$

Expanding into spatial and time components, recalling that $A^\mu = \left(\varphi, \vec{A}\right)$, we get:

$$\partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \quad \to \quad \nu = 0 \;\Rightarrow\; \Box\varphi + \partial_t(\partial_t \varphi + \vec{\nabla}\cdot\vec{A}) = 0$$

$$\nabla^2 \varphi = 0 \,; \tag{3}$$

$$\nu = i \;\Rightarrow\; \Box A^i - \partial_i \left(\partial_t \varphi + \vec{\nabla}\cdot\vec{A}\right) = 0$$

$$\Box A^i - \partial_i \dot{\varphi} = 0 \,, \tag{4}$$

where, in (3) and (4), we have imposed the Coulomb gauge condition, $\vec{\nabla}\cdot\vec{A} = 0$. We see that the $\nu = 0$ equation for φ is only a constraint equation, in the sense that φ has no equation of motion with a double time derivative, and therefore is not a dynamical field. Provided appropriate boundary conditions, it can therefore be solved for all times in terms of the other dynamical fields $\vec{A}$ by solving the constraint equation (3) and substituting it in (4).

Since (3) only fixes φ up to some (possibly time-dependent) integration constants, our gauge choice $\vec{\nabla}\cdot\vec{A} = 0$ does not completely fix the gauge. To fix this residual gauge freedom, we can impose $\varphi = 0$. To show this makes sense, let's say we start fem a gauge where $\varphi \neq 0$, and show it is possible to make a gauge transformation to the $\varphi = 0$ gauge. That is, if $\varphi = f(t)$ where f is an arbitrary function of time, we can make the gauge transformation $A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \lambda$ (where λ is an arbitrary scalar function) by choosing $\partial_t \lambda = -f(t)$ (note that we know this is always possible since this is a gauge transformation and so $\tilde{A}_\mu$ still satisfies the equations of motion, in particular $\nabla^2 \tilde{\varphi} = \nabla^2(\varphi - \partial^t \lambda) = 0$) and $\partial^i \partial_i \lambda = 0$, which always has a solution. Making this gauge transformation, we remain in the Coulomb gauge, since then $\vec{\nabla}\cdot\vec{\tilde{A}} = \vec{\nabla}\cdot\vec{A} - \nabla^2\lambda = \vec{\nabla}\cdot\vec{A} = 0$

Now, looking at the operator

$$P_{ij} = \delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2} \,, \tag{5}$$

we can show that this is indeed a projection operator if we show $P_{ij} P_{jk} = P_{ik}$ (note that here, $(\nabla^2)^{-1}$ is defined as the Green's function for the Laplacian). Therefore:

$$\begin{aligned} P_{ij} P_{jk} &= \delta_{ij}\delta_{jk} - 2\delta_{ij} \frac{\nabla_i \nabla_k}{\nabla^2} + \frac{\nabla_i \nabla_j}{\nabla^2} \frac{\nabla_j \nabla_k}{\nabla^2} \\ &= \delta_{ik} - 2\frac{\nabla_i \nabla_k}{\nabla^2} + \frac{\nabla_i \nabla^2 \nabla_k}{\nabla^2 \nabla^2} \\ &= P_{ik} \,. \end{aligned} \tag{6}$$

We can read off the propagator from Maxwell's Lagrangrian in the Coulomb gauge:

$$L_{Coulomb} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} A_i \Box A^i \,. \tag{7}$$

Therefore, the propagator is:

$$D_F(k)^{Coulomb}_{ij} = \frac{\delta_{ij}}{k^2 - i\epsilon} \,. \tag{8}$$

The Coulomb gauge fixed propagator that was defined in class is

$$\tilde{D}_F(k)_{ij}^{Coulomb} = \frac{1}{k^2 - i\epsilon}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \tag{9}$$

To argue that $D_F(k)$ and $\tilde{D}_F(k)$ are indeed equivalent at tree level for physical processes, we only need to realize that $\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$ being a projection operator, the role it plays in the propagator is to project out the two physical degrees of freedom of the photon out of the three dimensional vectors it acts on. Since at tree-level there are no virtual particles in loops and all the states are on-shell, the projection operator acts trivially.

Exercise 2

We'll solve this question in some detail highlighting several general features of the calculation. Consider the potential

$$V(q) = \frac{1}{2}m\omega^2 q^2 \tag{10}$$

This is the harmonic oscillator, one of the two solvable problems in quantum mechanics, the second one being the hydrogen atom. All paths $q(t)$ have the property that

$$q(0) = q_i, \qquad q(T) = q_f \tag{11}$$

In this case, the classical path will play a role in the evaluation of the path integral. We will expand $q(t)$ about the classical path $q_c(t)$, which is defined as the path obeying

$$\ddot{q}_c + \omega^2 q_c = 0, \quad q_c(0) = q_i, \quad q_c(T) = q_f \tag{12}$$

This equation is easy to solve, all solutions must include $\sin\omega t$ and $\cos\omega t$ with the general solution being a linear combination. However it is much nicer to choose $\sin\omega t$ and $\sin\omega(T-t)$[1]. Now,

$$q_c(t) = \frac{1}{\sin\omega T}\left(q_f \sin\omega t + q_i \sin\omega(T-t)\right) \tag{13}$$

which solves (12) and satisfies the boundary conditions. The action for the classical path is given by

$$S[q_c] = \frac{1}{2}m\int_0^T dt(\dot{q}_c^2 - \omega^2 q_c^2) \tag{14}$$

Integrating this by parts we get,

$$S[q_c] = \frac{1}{2}m\left[q_c\dot{q}_c\right]_0^T - \frac{1}{2}m\int_0^T dt\; q_c(\ddot{q}_c + \omega^2 q_c) \tag{15}$$

where the last integral vanishes since q_c obeys the equation of motion (12). Substituting the boundary conditions in (15) we get

$$S[q_c] = \frac{1}{2}m\left(q_f\dot{q}_c(T) - q_i\dot{q}_c(0)\right) \tag{16}$$

Using the explicit solution (13), we get

$$S[q_c] = m\omega\frac{-2q_f q_i + (q_f^2 + q_i^2)\cos\omega T}{2\sin\omega T} \tag{17}$$

Now comes the clever bit, we can write a general path as

$$q(t) = q_c(t) + f(t), \qquad f(0) = f(T) = 0 \tag{18}$$

For S[q] quadratic in q then

$$S[q] = S[q_c] + S[f] \tag{19}$$

is exact since $S[q]$ is stationary at $q = q_c$. (If you vary the action, you get the classical equations of motion - that is what the action comes from) i.e. note that there are no linear terms in f. This is analogous to expanding a function around a minimum, where the first terms in the expansion will be the value of the function at the minimum and a term quadratic in the deviation from the minimum.

We have also assumed that f is small, meaning that only paths close to the classical path will contribute to the integral. Furthermore, we can assume

$$d[q] = d[f] \tag{20}$$

[1] These are actually identical in the case $\omega T = n\pi$ for some integer n. In this case there is no classical solution unless $q = (-1)^n q_0$. This special case is ignored here.

which is true for ordinary integrals, namely that $d(x+a) = dx$ for constant a (this is also makes sense in terms of the more formal definition of the measure of the path integral given in lectures). We can now write that in this particular example,

$$K(q_f, q_i; T) = \int d[q] e^{iS[q]} = e^{iS[q_c]} \int d[f] e^{iS[f]} \tag{21}$$

Note that the integral is independent of q_i, q_f, thus all the dependence on the initial and final points is contained in the pre factor. There are various ways to derive the second factor: we use the one which is potentially useful later on in AQFT. Expand $f(t)$ in terms of a convenient complete set. We use a Fourier sine series for f:

$$f(t) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{T}} \sin \frac{n\pi t}{T} \tag{22}$$

The sine functions form an orthogonal basis for functions vanishing at $t = 0$ and $t = T$. We can write, integrating by parts, noting $f(0) = f(T) = 0$ and using orthonormality,

$$S[f] = -\frac{1}{2} m \int_0^T dt f(\ddot{f} + \omega^2 f) = \frac{1}{2} m \sum_n a_n^2 \left(\frac{n^2 \pi^2}{T^2} - \omega^2 \right) \tag{23}$$

We assume the relation

$$d[f] = C \prod_{n=1}^{\infty} da_n \tag{24}$$

where C is a normalisation constant.[2] We are now in a position to express the integral in the form

$$\int d[f] e^{iS[f]} = C \prod_{n=1}^{\infty} \int da_n e^{i\frac{m}{2} a_n^2 \left(\frac{n^2\pi^2}{T^2} - \omega^2 \right)} \tag{25}$$

The basic interval (solved by rotating the contour) is

$$\int_{-\infty}^{\infty} dy e^{\frac{i}{2} \lambda y^2} = \sqrt{\frac{2\pi i}{\lambda}} \tag{26}$$

It is convenient to write

$$\int d[f] e^{iS[f]} = C_0 \prod_{n=1}^{\infty} \frac{1}{\sqrt{1 - \frac{\omega^2 T^2}{n^2 \pi^2}}} \tag{27}$$

where we have absorbed constants like $(\prod_{n=1}^{\infty} n)^{-1}$ into C_0. This factor is divergent, but does not depend on any of the critical parameters. (We will of course not talk about infinities here). In the free case $\omega = 0$, this product is equal to one and we are left with

$$C_0 = \sqrt{\frac{m}{2\pi i T}} \tag{28}$$

which fixes the normalisation. This result was derived for the free theory in lectures. See also Hugh Osborn's AQFT notes for details http://www.damtp.cam.ac.uk/user/ho/Notes.pdf. What can we say about this infinity product? It can be shown that

$$\prod_{n=1}^{\infty} \left(1 - \frac{\omega^2 T^2}{n^2 \pi^2} \right) = \frac{\sin \omega T}{\omega T} \tag{29}$$

(Note that both sides have the same zeros as functions of ωT. Furthermore, they both go to one as $\omega T \to 0$. Several other observations show that both sides have the same behaviour and are actually identical.)
Ultimately,

$$\int d[f] e^{iS[f]} = \sqrt{\frac{m\omega}{2\pi i \sin \omega T}} \tag{30}$$

which is a nice everyday function. The overall result, in all its glory, is the following:

$$K(q_f, q_i; T) = \sqrt{\frac{m\omega}{2\pi i \sin \omega T}} e^{im\omega \frac{-2q_f q_i + (q_f^2 + q_i^2)\cos \omega T}{2 \sin \omega T}} \tag{31}$$

To check this, note that there are eigenfunctions $|n\rangle$ of $\hat{H}$ with

$$E_n = \left(n + \frac{1}{2} \right) \omega, \qquad \hat{1} = \sum_n |n\rangle\langle n| \tag{32}$$

[2] Thinking about the formal definition of the measure of a path integral (lecture notes/almost any book on QFT that does path integrals, e.g. Zee), if we consider $t_r = \epsilon r$, with $\epsilon = T/(n+1)$, then the transformation from $f(t_r) : r = 1, \cdots, N$ to $a_n : n = 1, \cdots, N$ is an orthogonal transformation so that $\prod_r df(t_r) = \prod_n da_n$

A formula that can be obtained by using standard quantum mechanics is

$$K(q_f, q_i; T) = \sum_n \psi_n(q_f)\psi_n^\star(q_i)e^{-i(n+\frac{1}{2})\omega T} \tag{33}$$

In the situation where $T = -i\tau,\ \tau \to \infty$

$$2i\sin\omega T \to e^{\omega\tau}, \quad 2\cos\omega T \to e^{\omega\tau} \tag{34}$$

We find that

$$K(q_f, q_i; -i\tau) \xrightarrow{\tau\to\infty} \sqrt{\frac{m\omega}{\pi}}e^{-\frac{1}{2}\omega\tau}e^{-\frac{1}{2}m\omega(q_f^2+q_i^2)} = \psi_0(q_f)\psi_0^\star(q_i)e^{-\frac{1}{2}\omega\tau} \tag{35}$$

In this special case, the results match.

Few Comments:

1) Generally speaking, integrals of the form $\int dx\ e^{i\lambda x^2}$ are rather ill-defined because they do not converge absolutely. It is much better to consider integrals of the form $\int dx\ e^{-\lambda x^2}$ for $\lambda > 0$. The same thing happens with path integrals; note that

$$S[q] = \int_0^T dt\ \left(\frac{1}{2}m\dot{q}^2 - V(q)\right) \tag{36}$$

is normally a real quantity. But in order to obtain well-defined integrals, we consider an analytic continuation of time

$$t \to -i\tau, \quad T \to -i\tau_1 \tag{37}$$

so that the path integral now becomes

$$\langle q|exp(-\hat{H}\tau)|q_0\rangle = \int d[q]e^{-\int_0^{\tau_1} d\tau\left(\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2+v(q)\right)} \tag{38}$$

The point is that the RHS integral above has a rigorous definition for a wide class of potentials V, subject to the requirement that V is bounded from below.

2) Path integrals provide a method for making non-perturbative approximations; we will show this for the example of tunnelling. A widely know example form quantum mechanics is that particle can tunnel through a potential barrier. Consider a potential $V(q)$ with 2 minima at q_0 and q_1, i.e. shaped like a "W". We want to calculate the amplitude to get from q_0 at time t_0 to q_1 at time t_1, eventually taking the limits $t_0 \to -\infty$ and $t_1 \to \infty$. So use the path integral to valuate

$$\langle q_1|exp(-i\hat{H}(t_1 - t_0)|q_0\rangle \tag{39}$$

One way of proceeding with there path integrals is to expand around a classical path

$$q(t) = q_c(t) + f(t) \tag{40}$$

where the classical path $q_c(t)$ satisfies the classical equations and given boundary conditions. In general, there will not necessarily be a classical path; here this is in the case when the total energy is smaller than the maximum of the potential between q_0 and q_1. We make use of analytic continuation $t \to -i\tau$, such that

$$iS[q] = -\int_{\tau_0}^{\tau_1} d\tau\left(\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 + V(q)\right) \tag{41}$$

and we are interested in the limit $t_0 \to -\infty$ and $t_1 \to \infty$. (Note that this actually means that we choose a different contour in the complex place to evaluate the integral. This is a method commonly used to evaluate integrals over analytic functions and we have infact already used it in the calculation above. One makes use of this in the method of steepest descent for e.g.) The classical equation for $q_c(\tau)$ is now

$$-m\frac{d^2}{d\tau^2}q_c + V'(q_c) = 0 \tag{42}$$

which we integrate once to get

$$-\frac{1}{2}m\left(\frac{dq_c}{d\tau}\right)^2 + V(q_c) = E \tag{43}$$

This is similar to classical mechanics, but with one sign flipped because of our funny change in time. We want a situation in which as $\tau \to \pm\infty, q(\tau) \to q_0$ or $q(\tau) \to q_1$, where $V(q_0) = V(q_1) = 0$. But if it is smoothly going to these points we must also have

$$\left.\frac{dq_c}{d\tau}\right|_{q=q_0} = \left.\frac{dq_c}{d\tau}\right|_{q=q_1} = 0 \quad \Rightarrow\ E = 0 \tag{44}$$

In this situation we can actually solve this, assuming $q_1 > q_0$ and therefore taking the positive square root, we get

$$\frac{dq_c}{d\tau} = \sqrt{\frac{2V(q_c)}{m}} \tag{45}$$

Now substitute this in to evaluate $iS[q_c]$,

$$iS[q_c] = -2\int_{-\infty}^{\infty} d\tau\; V(q_c) = -\int_{q_0}^{q_1} dq\; \sqrt{2mV(q)} \tag{46}$$

because $d\tau = \frac{dq\sqrt{m}}{\sqrt{2V(q)}}$ for this solution. As we have seen in the case of the harmonic oscillator, the dependence of the path integral on the initial and final points is contained in $e^{iS[q_c]}$, so the tunnelling amplitude will be proportional to

$$e^{-\int_{q_0}^{q_1} dq\; \sqrt{2mV(q)}} \tag{47}$$

This is an exponential suppression which is a real quantity, non-zero in all cases, also known as the Gamov factor. The path integral calculation gives the same result as WKB in a relatively simple way.

Exercise 3

As the question says, this is bookwork and can be found in most QFT textbooks. For a free version, see pages 18-24 of Hugh Osborn's advanced quantum field theory lecture notes, available at: http://www.damtp.cam.ac.uk/user/examples/3P5a.pdf. Let us begin by calculating a so-called n-point correlation function. This is defined as

$$\lim_{T\to\infty} \langle 0|T\phi(x_1)\dots\phi(x_n)\exp(i\int_{-T}^{T} Hdt)|0\rangle = \frac{\int \mathcal{D}\phi\; \phi(x_1)\dots\phi(x_n)e^{iS}}{\int \mathcal{D}\phi\; e^{iS}}\,. \tag{48}$$

We take the ratio because the path integral is defined up to a normalisation constant.
Now, let us express this correlation function in terms of the sourced path integral

$$Z[J] \equiv \int \mathcal{D}\phi e^{iS+iJ\phi}\,, \tag{49}$$

to get

$$\langle 0|T\{\phi(x_1)\dots\phi(x_n)\}|0\rangle = \left.\frac{\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\dots\left(-i\frac{\delta}{\delta J(x_n)}\right)Z[J]}{Z[0]}\right|_{J=0}, \tag{50}$$

where we have also used the short-hand notation

$$T\{\phi(x_1)\dots\phi(x_n)\} \equiv \lim_{T\to\infty} T\phi(x_1)\dots\phi(x_n)\exp(i\int_{-T}^{T} Hdt)\,. \tag{51}$$

Let us begin by studying free scalar theory. Then

$$Z[J] = Z[0]\exp\left(\frac{i}{2}JA^{-1}J\right)\,, \tag{52}$$

where "$JA^{-1}J$" is shorthand for

$$\int dx\; dy\; J(x)A^{-1}(x,y)J(y)\,, \tag{53}$$

and $A^{-1}(x,y)$ is the Green's function from lectures

$$A^{-1}(x,y) = \frac{1}{-\Box + m^2}(x,y)\,. \tag{54}$$

This allows us to calculate the free two-point correlation function

$$\begin{aligned}\langle 0|T\{\phi(x)\phi(y)\}|0\rangle &= \left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)\exp\left(\frac{i}{2}JA^{-1}J\right)\Big|_{J=0}\\ &= (-i)^2\left[iA^{-1}(x,y) + \int du\; iA^{-1}(x,u)J(u)\int dv\; iA^{-1}(y,v)J(v)\right]\exp\left(\frac{i}{2}JA^{-1}J\right)\Big|_{J=0}\\ &= (-i)^2\, iA^{-1}(x,y)\\ &= -iA^{-1}(x,y)\,,\end{aligned} \tag{55}$$

where we see that because the path integral is evaluated at $J=0$, no free Js can survive after differentiation, except for in the exponential. Thus, we see that the 2-point correlation function is just $(-i)$ times the Green's function and we can express it diagramatically by a line joining points x and y.
Similarly, for the 4-point correlation function

$$\begin{aligned}\langle 0|T\left\{\phi(x)\phi(y)\phi(z)\phi(w)\right\}|0\rangle &= \left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)\left(-i\frac{\delta}{\delta J(z)}\right)\left(-i\frac{\delta}{\delta J(w)}\right)\exp\left(\frac{i}{2}JA^{-1}J\right)\Bigg|_{J=0} \\ &= (-i)^4\left[iA^{-1}(x,y)\ iA^{-1}(z,w)+iA^{-1}(x,z)\ iA^{-1}(y,w)\right. \\ &\left.+iA^{-1}(x,w)\ iA^{-1}(y,z)\right]\exp\left(\frac{i}{2}JA^{-1}J\right)\Bigg|_{J=0} \\ &= \left(-iA^{-1}(x,y)\right)\left(-iA^{-1}(z,w)\right)+\left(-iA^{-1}(x,z)\right)\left(-iA^{-1}(y,w)\right) \\ &+\left(-iA^{-1}(x,w)\right)\left(-iA^{-1}(y,z)\right)\end{aligned} \tag{56}$$

We can draw diagrams for these terms again by joining the three pairs out of x,y,z,w by lines and for each line between the points, say x,y we associate a propagator $-iA^{-1}(x,y)$.
Let us now study interactions before moving to momentum space (which we will show is relevant for scattering). The interacting 2-point correlation function can also be written as

$$\langle 0|T\left\{\phi(x)\phi(y)\right\}|0\rangle = \frac{\left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)Z_{int}[J]}{Z_{int}[0]}\Bigg|_{J=0}, \tag{57}$$

where

$$Z_{int}[J]=\int\mathcal{D}\phi e^{iS_{free}-iV(\phi)+iJ\phi}=\exp\left[-iV\left(-i\frac{\delta}{\delta J}\right)\right]Z_{free}[J]. \tag{58}$$

For ϕ^4 theory the potential term is $V(\phi)=\frac{\lambda}{4!}\phi^4$ giving

$$\exp\left[-iV\left(-i\frac{\delta}{\delta J}\right)\right]=\exp\left[-i\int du\frac{\lambda}{4!}\left(-i\frac{\delta}{\delta J(u)}\right)^4\right], \tag{59}$$

which we Taylor expand

$$\exp\left[-iV\left(-i\frac{\delta}{\delta J}\right)\right]=1-i\int du\frac{\lambda}{4!}\left(-i\frac{\delta}{\delta J(u)}\right)^4+\dots. \tag{60}$$

Now we can calculate the numerator of (57)

$$\begin{aligned}\left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)Z_{int}[J]\Bigg|_{J=0} &= \left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)\left(1-i\int du\frac{\lambda}{4!}\left(-i\frac{\delta}{\delta J(u)}\right)^4+\dots\right)Z_{free}[J]\Bigg|_{J=0} \\ &= Z_{free}[0]\left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)\left(1-i\int du\frac{\lambda}{4!}\left(-i\frac{\delta}{\delta J(u)}\right)^4+\dots\right) \\ &\times\exp\left[\frac{i}{2}JA^{-1}J\right]\Bigg|_{J=0} \\ &= Z_{free}[0]\left[-iA^{-1}(x,y)-i\lambda\int dz\left(-iA^{-1}(x,z)\right)\left(-iA^{-1}(z,z)\right)\left(-iA^{-1}(y,z)\right)\right. \\ &\left.-i\lambda\left(-iA^{-1}(x,y)\right)\int dz\left(-iA^{-1}(z,z)\right)\left(-iA^{-1}(z,z)\right)+\dots\right]\end{aligned} \tag{61}$$

Again, we can express these terms diagramatically to find again the first term corresponding to a propagator from x to y which gives a contribution $-iA^{-1}(x,y)$. The second term corresponds to a propagator with a loop attached. Each propagator gives a contribution $-iA^{-1}$ while the vertex gives the contribution $-i\lambda$. The final term is disconnected: it is a propagator from x to y again and a vacuum diagram.
David Tong's lecture notes (section 3.7) discuss that these diagrams factor as

$$\left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)Z_{int}[J]\Bigg|_{J=0}=(\text{connected diagrams})\times(\text{vacuum bubbles}). \tag{62}$$

Returning to (57) we see that the denominator will just give the vacuum bubbles so that the whole two-point correlation function is just

$$\langle 0|T\left\{\phi(x)\phi(y)\right\}|0\rangle=\sum(\text{connected diagrams}). \tag{63}$$

Finally, we want to go to momentum space which corresponds to calculating

$$\int dx\ dy\ e^{ipx} e^{-iqy} \langle 0| T\left\{\phi(x)\phi(y)\right\}|0\rangle\,. \tag{64}$$

It is sufficient to study the free correlation function to understand how the Feynman rules are modified. We find

$$\begin{aligned}
\int dx\ dy\ e^{ipx} e^{-iqy} \langle 0| T\left\{\phi(x)\phi(y)\right\}|0\rangle &= \int dx\ dy\ e^{ipx} e^{-iqy} - iA^{-1}(x,y) \\
&= \int dx\ dy\ e^{ipx} e^{-iqy} \int \frac{dk}{(2\pi)^4} \frac{-i}{k^2+m^2} e^{-ik(x-y)} \\
&= \int \frac{dk}{(2\pi)^4}\ dx\ dy\ \frac{-i}{k^2+m^2} e^{ix(p-k)} e^{iy(k-q))} \\
&= \int dk\ dy \frac{-i}{k^2+m^2} \delta(p-k) e^{iy(k-q)} \\
&= (2\pi)^4 \int dk \frac{-i}{k^2+m^2} \delta(p-k)\delta(k-q) \\
&= \frac{-i}{p^2+m^2} (2\pi)^4 \delta(p-q)\,.
\end{aligned} \tag{65}$$

Thus, we see that now overall momentum conservation is imposed $(2\pi)^4 \delta(p_{initial} - p_{final})$ and we use the momentum representation of the propagator

$$\frac{-i}{p^2+m^2}\,. \tag{66}$$

Finally, let us turn to scattering amplitudes.

$$\langle q|S|p\rangle = \sqrt{4E_q E_p} \langle 0| a_q S a_p^\dagger |0\rangle\,. \tag{67}$$

We can then write

$$a_p^\dagger |0\rangle \approx \int dx \phi(x) e^{-ipx} |0\rangle \tag{68}$$

and so we see that the scattering amplitude will give

$$\langle q|S|p\rangle \approx \int dx dy e^{-ipy} e^{iqx} \langle 0| T\left\{\phi(x)\phi(y)\right\}|0\rangle\,, \tag{69}$$

so that n-particle scattering amplitudes are just Fourier transforms of n-point correlation functions. Thus, we summarise that we get the following Feynman rules. Draw all *connected* diagrams with required number of external lines and label external momenta where only 4-point vertices are to be included. Associate

- for every internal propagator a factor of $\frac{-i}{p^2+m^2}$,
- for every vertex a factor of $-i\lambda$,
- for every internal loop an undetermined momentum k and an integral with measure $\int \frac{d^4k}{(2\pi)^4}$,
- impose momentum conservation at every vertex and overall by $(2\pi)^4 \delta(p_{in} - p_{out})$.

Non-examinable details: LSZ reduction formula

Scattering amplitudes can be more carefully calculated using the **LSZ reduction formula** which states that

$$\begin{aligned}
\langle q_1, q_2, \ldots q_n|S|p_1, p_2 \ldots p_m\rangle = i^{n+m} &\int d^4x_1 e^{iq_1x_1}\left(-\Box_{x_1}+m^2\right) \int d^4x_2 e^{iq_2x_2}\left(-\Box_{x_2}+m^2\right) \\
\times &\int \ldots d^4x_n e^{iq_nx_n}\left(-\Box_{x_n}+m^2\right) \int d^4y_1 e^{ip_1y_1}\left(-\Box_{y_1}+m^2\right) \int d^4y_2 e^{ip_2y_2}\left(-\Box_{y_2}+m^2\right) \\
\ldots &\int d^4y_m e^{ip_my_m}\left(-\Box_{y_m}+m^2\right) \langle 0|T\left\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\phi(y_1)\phi(y_2)\ldots\phi(y_m)\right\}|0\rangle\,,
\end{aligned} \tag{70}$$

which can be further simplified by Fourier-transforming the $(n+m)$-point correlation function

$$\begin{aligned}
\langle 0|T\left\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\phi(y_1)\phi(y_2)\ldots\phi(y_m)\right\}|0\rangle &= \int \frac{1}{(2\pi)^{n+m}} d^4k_1\ d^4k_2 \ldots d^4k_n\ d^4l_1\ d^4l_2 \ldots\ d^4l_m \\
\times e^{-ik_1x_1} e^{-ik_2x_2} \ldots e^{-il_1y_2} e^{-il_my_m} \ldots e^{-il_my_m} &\tilde{G}^{n+m}(k_1, k_2, \ldots, k_n, l_1, l_2, \ldots, l_m)\,.
\end{aligned} \tag{71}$$

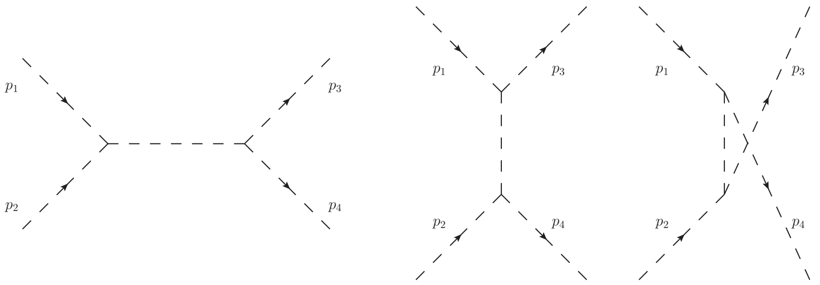
Then the LSZ reduction formula becomes

$$\begin{aligned}
\langle q_1, q_2, \ldots q_n|S|p_1, p_2 \ldots p_m\rangle = i^{n+m} &\left(q_1^2+m^2\right)\left(q_2^2+m^2\right)\ldots\left(q_n^2+m^2\right)\left(p_1^2+m^2\right)\left(p_2^2+m^2\right)\ldots\left(p_m^2+m^2\right) \\
\times &\tilde{G}^{n+m}(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_m)\,.
\end{aligned} \tag{72}$$

This is just saying that we calculate the $(n+m)$-point correlation functions in momentum space but don't include the propagator for the external legs (these are exactly cancelled by the $\left(q^2+m^2\right)$ factors).

Exercise 4

We can do the same thing as in exercise three to obtain the Feynman rules. The only change now is that there is a 3-point vertex which gives a factor of $-i\lambda$.
There are three tree-level diagrams



They have two vertices and one internal line whose momentum, k, is completely fixed by momentum conservation. Let us label initial momenta p_1, p_2 and final momenta p_3, p_4. Then for the first diagram the internal momentum is

$$k = p_1 + p_2 \,, \tag{73}$$

while for the second diagram it would be

$$k = p_1 - p_3 \,, \tag{74}$$

and for the third it is

$$k = p_1 - p_4 \,. \tag{75}$$

We then have the amplitudes

$$\begin{aligned} A_1 &= (2\pi)^4 \,\delta\left(p_1 + p_2 - p_3 + p_4\right) (-i\lambda)^2 \frac{-i}{\left(p_1 + p_2\right)^2 + m^2} \\ &= i\lambda^2 \,(2\pi)^4 \,\delta\left(p_1 + p_2 - p_3 + p_4\right) \frac{1}{\left(p_1 + p_2\right)^2 + m^2} \,, \\ A_2 &= i\lambda^2 \,(2\pi)^4 \,\delta\left(p_1 + p_2 - p_3 + p_4\right) \frac{1}{\left(p_1 - p_3\right)^2 + m^2} \,, \\ A_3 &= i\lambda^2 \,(2\pi)^4 \,\delta\left(p_1 + p_2 - p_3 + p_4\right) \frac{1}{\left(p_1 - p_4\right)^2 + m^2} \,. \end{aligned} \tag{76}$$

$$A_1 + A_2 + A_3 = A_{total} \equiv i\,(2\pi)^4 \,\delta(p_1 + p_2 - p_3 + p_4) M \,. \tag{77}$$

Let us define the Mandelstam variables

$$\begin{aligned} s &\equiv \left(p_1 + p_2\right)^2 \,, \\ t &\equiv \left(p_1 - p_3\right)^2 \,, \\ u &\equiv \left(p_1 - p_4\right)^2 \,. \end{aligned} \tag{78}$$

Then, the total amplitude can be written as

$$A_{total} = i\lambda^2 \,(2\pi)^4 \,\delta(p_1 + p_2 - p_3 + p_4) \left[\frac{1}{s + m^2} + \frac{1}{t + m^2} + \frac{1}{u + m^2}\right] . \tag{79}$$

Finally, the differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{p_3}|}{|\vec{p_1}|} \frac{|M|^2}{64\pi^2 E_{com}^2} = \frac{|\vec{p_3}|}{|\vec{p_1}|} \frac{\lambda^4}{64\pi^2 s} \left[\frac{1}{s + m^2} + \frac{1}{t + m^2} + \frac{1}{u + m^2}\right]^2 . \tag{80}$$

For further details see, for example, Srednicki, Quantum Field Theory, http://web.physics.ucsb.edu/ mark/qft.html