Quantum Field Theory
Example Sheet 4
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Exercise 1

This process can be found in chapter 5 of Peskin and Schroeder but using different conventions for the metric and gamma matrices. Muon-electron scattering has one Feynman diagram

Labelling the internal momentum of the photon as q, the matrix element for this process is:

$$\mathcal{M} = \frac{ie^2}{q^2} \bar{u}(p_1')\gamma^\mu u(p_1)\bar{u}(p_2')\gamma_\mu u(p_2) \tag{1}$$

Ignoring the spin in the final state means we sum over r', s'. We also average over the initial spins which requires summing over r, s and dividing by 4, since they are spin-1/2 particles. The squared amplitude is then

$$\frac{1}{4}\sum_{r,r',s,s'} |\mathcal{M}|^2 = \frac{e^4}{4q^4} \sum_{r,r',s,s'} \bar{u}(p_1')\gamma^\mu u(p_1)\bar{u}(p_2')\gamma_\mu u(p_2)\ \bar{u}(p_2)\gamma_\nu u(p_2')\bar{u}(p_1)\gamma^\nu u(p_1') \tag{2}$$

Using the spin sum relations gives

$$\frac{1}{4}\sum_{r,r',s,s'} |\mathcal{M}|^2 = \frac{e^4}{4q^4} \mathrm{tr}\left[(\not{p}_1' + im_e)\gamma^\mu(\not{p}_1 + im_e)\gamma^\nu\right] \mathrm{tr}\left[(\not{p}_2' + im_\mu)\gamma_\mu(\not{p}_2 + im_\mu)\gamma_\nu\right] \tag{3}$$

Where m_e, m_μ are the masses of the electron and muon. We can then proceed to evaluate the expression by taking the trace of the gamma matrices. Using the formulae for traces of two and four gamma matrices on the previous example sheets (the trace of an odd number is zero) gives

$$\begin{aligned} \mathrm{tr}\left[(\not{p}_1' + im_e)\gamma^\mu(\not{p}_1 + im_e)\gamma^\nu\right] &= p_{1\rho}p'_{1\sigma}\mathrm{tr}\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right] - m_e^2\mathrm{tr}(\gamma^\mu\gamma^\nu) \\ &= 4[{p_1'}^{\mu}{p_1}^{\nu} + {p_1'}^{\nu}{p_1}^{\mu} - p_1\cdot p_1' g^{\mu\nu} - m_e^2 g^{\mu\nu}] \end{aligned} \tag{4}$$

and similarly for the muon. Since $m_\mu \simeq 106$ MeV and $m_e \simeq 0.5$ MeV, we can neglect the mass of the electron. The error from doing so is less than the error from missing higher order diagrams. Taking the product of the two traces and simplifying gives

$$\frac{1}{4}\sum_{r,r',s,s'} |\mathcal{M}|^2 = \frac{32e^4}{4q^4}\left[(p_1'\cdot p_2')(p_1\cdot p_2) + (p_1\cdot p_2')(p_1'\cdot p_2) + m_\mu^2(p_1'\cdot p_1)\right] \tag{5}$$

To proceed further we need to simplify the kinematics. We work in the centre of mass frame so that the particles have spatial momentum only in the z direction. The initial and final momenta are:

$$p_1 = (k, k\hat{z}), \quad p_1' = (k, \mathbf{k}), \quad p_2 = (E, -k\hat{z}), \quad p_2' = (k, -\mathbf{k}) \tag{6}$$

Where we have used conservation of momentum. We will require the following inner products:

$$\begin{aligned} p_1\cdot p_2 &= p_1'\cdot p_2' = -k(E+k) \\ p_1\cdot p_1' &= -k^2(1-\cos\theta) \\ p_1'\cdot p_2 &= p_1\cdot p_2' = -k(E + k\cos\theta) \\ q^2 &= -2p_1\cdot p_1' = -2k^2(1-\cos\theta). \end{aligned} \tag{7}$$

Which give

$$\frac{1}{4}\sum_{r,r',s,s'} |\mathcal{M}|^2 = \frac{8e^4}{4(1-\cos\theta)^2}\left[(E+k\cos\theta)^2 + (E+k)^2 - m_\mu^2(1-\cos\theta)\right] \tag{8}$$

The spin-averaged differential cross section in the CM frame is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{4}\sum_{\text{spins}}\frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2} \tag{9}$$

$$= \frac{\alpha^2}{2E_{CM}^2(1-\cos\theta)^2}\left[(E+k\cos\theta)^2+(E+k)^2-m_\mu^2(1-\cos\theta)\right] \tag{10}$$

A further approximation can be made in the high energy limit. We can set $m_\mu = 0$ which reduces the result to

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{2E_{CM}^2(1-\cos\theta)^2}\left[4+(1+\cos\theta)^2\right] \tag{11}$$

Exercise 2

The two relevant diagrams for this case are:

Now given the Feynman rules for QED, we can write down the amplitude for this process

$$\mathcal{M} = ie^2\left[\left[\frac{\bar{u}(k')\gamma^\mu u(k)\bar{u}(p')\gamma_\mu u(p)}{(k'-k)^2}\right]-\left[\frac{\bar{u}(p')\gamma^\mu u(k)\bar{u}(k')\gamma_\mu u(p)}{(p'-k)^2}\right]\right] \tag{12}$$

To make life easier, lets call the first piece $A(k',k)$ and the second piece $A(p',k)$. Ignoring the spin in the final state means we sum over r', s'. We also average over the initial spins which requires summing over r, s and dividing by 4, since they are spin-1/2 particles. The squared amplitude is then

$$\frac{1}{4}\sum_{r,r',s,s'}|\mathcal{M}|^2 = \frac{e^4}{4}\sum_{r,r',s,s'}\left[|A(k',k)|^2+|A(p',k)|^2-2\text{Re}[A(k',k)*A(p',k)]\right] \tag{13}$$

Now we'll evaluate each piece separately. The first one is

$$|A(k',k)|^2 = \frac{1}{(k'-k)^4}[\bar{u}(k')\gamma^\mu u(k)u(\bar{k})\gamma^\nu u(k')][\bar{u}(p')\gamma_\mu u(p)\bar{u}(p)\gamma_\nu u(p')] \tag{14}$$

now using Equations (3) and (4) from Question 1 and assuming that we are interested in the relativistic limit of the cross section, in which case the mass of the electron may be neglected in comparison to the momenta, we get

$$\sum_{r,r',s,s'}|A(k',k)|^2 = \frac{32}{(k'-k)^4}\left[(k\cdot p)(k'\cdot p')+(k\cdot p')(k'\cdot p)\right] \tag{15}$$

Simillarly, the second piece is

$$\sum_{r,r',s,s'}|A(p',k)|^2 = \frac{32}{(p'-k)^4}\left[(k\cdot p)(k'\cdot p')+(k\cdot k')(p\cdot p')\right] \tag{16}$$

The third piece is slightly more involved and is of the form

$$\sum_{r,r',s,s'}A(k',k)*A(p',k) = \sum_{r,r',s,s'}\bar{u}(k')\gamma^\mu u(k)\bar{u}(p')\gamma_\mu u(p)\bar{u}(k)\gamma^\nu u(p')\bar{u}(p)\gamma_\nu u(k') \tag{17}$$

But in the massless electron limit and again using Equations (3) and (4) from Question 1, we can simplify this down to

$$\sum_{r,r',s,s'}A(k',k)*A(p',k) = \text{tr}(\not{k}'\gamma^\mu\not{k}\gamma^\nu\not{p}'\gamma_\mu\not{p}\gamma_\nu) \tag{18}$$

$$= -32(k\cdot p)(k'\cdot p') \tag{19}$$

Now introducing the Mandelstam variables

$$s \equiv (k+p)^2 = (k'+p')^2 \tag{20}$$

$$t \equiv (k'-k)^2 = (p'-p)^2 \tag{21}$$

$$u \equiv (p'-k)^2 = (k'-p)^2 \tag{22}$$

and working in the centre of mass frame in the relativistic limit, where we express everything in terms of the centre or mass energy E and the scattering angle θ, we have

$$k = E(1,0,0,1) \tag{23}$$
$$p = E(1,0,0,-1) \tag{24}$$
$$k' = E(1,\sin\theta,0,\cos\theta) \tag{25}$$
$$p' = E(1,-\sin\theta,0,-\cos\theta) \tag{26}$$

Hence,

$$k\cdot p = k'\cdot p' = 2E^2 = \frac{1}{2}s \tag{27}$$
$$k\cdot k' = p\cdot p' = 2E^2\sin^2\frac{\theta}{2} = -\frac{1}{2}t \tag{28}$$
$$k\cdot p' = p\cdot k' = 2E^2\cos^2\frac{\theta}{2} = -\frac{1}{2}u \tag{29}$$
$$(k'-k)^4 = (-2k'\cdot k)^2 = 16E^4\sin^4\frac{\theta}{2} = t^2 \tag{30}$$
$$(p'-k)^4 = u^2 \tag{31}$$

Hence assembling everything, we get

$$\frac{1}{4}\sum_{r,r',s,s'}|\mathcal{M}|^2 = \frac{e^4}{4}\left[\frac{s^2+u^2}{t^2}+\frac{2s^2}{tu}+\frac{s^2+t^2}{u^2}\right] \tag{32}$$
$$= \frac{e^4}{4}2\left[\frac{1}{\sin^4(\theta/2)}+1+\frac{1}{cos^4(\theta/2)}\right] \tag{33}$$

and the differential cross section defined in Equation (9) is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{4E^2}\left[\frac{1}{\sin^4(\theta/2)}+1+\frac{1}{cos^4(\theta/2)}\right] \tag{34}$$

where $\alpha = e^2/4\pi$.

Exercise 3

Exercise 4

Exercise 5

In the mostly + metric convention, the correct Lagrangian with a Yukawa interaction to consider is:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - i\bar{\psi}(\partial\!\!\!/ + m)\psi + g\phi\bar{\psi}\gamma^5\psi\,. \tag{35}$$

Therefore the interacting Hamiltonian density is:

$$\mathcal{H}_I = -g\phi_I\bar{\psi}_I\gamma^5\psi_I\,, \tag{36}$$

where the index I refers to the fields in the interaction picture.

The Feynman rules for this theory are:

- Propagators:

$$= \frac{-i}{k^2+\mu^2-i\epsilon} \quad \text{for every scalar field internal line;}$$
$$= \frac{ik\!\!\!/ - m}{k^2+m^2-i\epsilon} \quad \text{for every fermion internal line;} \tag{37}$$

- Vertices:

$$= \quad ig \quad \text{for every vertex;} \tag{38}$$

- External legs:

$$\begin{array}{llllll} = & 1 & \text{for every scalar initial state} & = & 1 & \text{for every scalar final state;} \\ = & u^s(p) & \text{for every fermion initial state} & = & \bar{u}^s(p) & \text{for every Fermion final state;} \\ = & \bar{v}^s(p) & \text{for every anti}-\text{fermion initial state} & = & v^s(p) & \text{for every anti}-\text{fermion final state;} \end{array} \tag{39}$$

- Integrate over each undetermined loop momentum: $\int \frac{d^4k}{(2\pi)^4}$;
- Impose momentum conservation at each vertex;
- Impose overall momentum conservation: $(2\pi)^4\delta^{(4)}\left(p_{in}-p_{out}\right)$;
- Figure out the overall sign of the diagram from the commutation of fermions;
- Multiply by the number of ways the diagram can be connected, and divide by $n!$, where n is the number of vertices in the diagram.

Fermion-Fermion scattering, $\psi\psi \to \psi\psi$, is described by the following two diagrams:
The first diagram gives:

$$2ig^2(2\pi)^4\delta^{(4)}\left(p+k-p'-k'\right)\frac{\bar{u}_a^{s'}(k')\gamma^5_{ab}u^s_b(k)\bar{u}^{r'}_c(p')\gamma^5_{cd}u^r_d(p)}{(p-p')^2+\mu^2}\,; \tag{40}$$

and the second diagram gives

$$-2ig^2(2\pi)^4\delta^{(4)}\left(p+k-p'-k'\right)\frac{\bar{u}_a^{r'}(p')\gamma^5_{ab}u^s_b(k)\bar{u}^{s'}_c(k')\gamma^5_{cd}u^r_d(p)}{(p-k')^2+\mu^2}\,, \tag{41}$$

where the factor of 2 sitting in front each diagram comes from the two ways there are to connect each one of them, and the roman indices a, b, c, d are the spinor indices (it is useful to keep track of them when squaring the matrix element, to spot the traces).
The matrix element is then given by:

$$i\mathcal{M} = 2\frac{ig^2}{2!}\left[\frac{\bar{u}_a^{s'}(k')\gamma^5_{ab}u^s_b(k)\bar{u}^{r'}_c(p')\gamma^5_{cd}u^r_d(p)}{(p-p')^2+\mu^2} - \frac{\bar{u}_a^{r'}(p')\gamma^5_{ab}u^s_b(k)\bar{u}^{s'}_c(k')\gamma^5_{cd}u^r_d(p)}{(p-k')^2+\mu^2}\right] \tag{42}$$

To find the differential cross-section for this process, we need to calculate $|\mathcal{M}|^2 = \mathcal{M}\left(\mathcal{M}\right)^*$, and will need to use the following complex conjugate identity for bi-spinor products:

$$\left(\bar{u}(p)\gamma^5u(k)\right)^* = \left(u^\dagger(p)\gamma^0\gamma^5u(k)\right)^* = u^\dagger(k)\left(\gamma^5\right)^\dagger\left(\gamma^0\right)^\dagger u(p) = -u^\dagger(k)\gamma^5\gamma^0u(p) = u^\dagger(k)\gamma^0\gamma^5u(p) = \bar{u}(k)\gamma^5u(p) \tag{43}$$

Averaging over initial spins and summing over final spin states, we obtain:

$$\begin{aligned} \frac{1}{2}\sum_s\frac{1}{2}\sum_r\sum_{s',r'}|\mathcal{M}|^2 = {} & \frac{1}{4}\sum_{\text{spins}}g^4\left[\frac{\bar{u}_a^{s'}(k')\gamma^5_{ab}u^s_b(k)\bar{u}^{r'}_c(p')\gamma^5_{cd}u^r_d(p)}{(p-p')^2+\mu^2} - \frac{\bar{u}_a^{r'}(p')\gamma^5_{ab}u^s_b(k)\bar{u}^{s'}_c(k')\gamma^5_{cd}u^r_d(p)}{(p-k')^2+\mu^2}\right] \\ & \times\left[\frac{\bar{u}_e^{s}(k)\gamma^5_{ef}u^{s'}_f(k')\bar{u}^{r}_g(p)\gamma^5_{gh}u^{r'}_h(p')}{(p-p')^2+\mu^2} - \frac{\bar{u}_e^{s}(k)\gamma^5_{ef}u^{r'}_f(p')\bar{u}^{r}_g(p)\gamma^5_{gh}u^{s'}_h(k')}{(p-k')^2+\mu^2}\right] \end{aligned} \tag{44}$$

$$\begin{aligned} = {} & \frac{g^4}{4}\left[\frac{(-1)\left(\not{k}'+im\right)_{fa}\gamma^5_{ab}(-1)\left(\not{k}+im\right)_{be}\gamma^5_{ef}(-1)\left(\not{p}'+im\right)_{hc}\gamma^5_{cd}(-1)\left(\not{p}+im\right)_{dg}\gamma^5_{gh}}{\left((p-p')^2+\mu^2\right)^2}\right. \\ & -\frac{(-1)\left(\not{p}'+im\right)_{ha}\gamma^5_{ab}(-1)\left(\not{k}+im\right)_{be}\gamma^5_{ef}(-1)\left(\not{k}'+im\right)_{fc}\gamma^5_{cd}(-1)\left(\not{p}+im\right)_{dg}\gamma^5_{gh}}{\left((p-p')^2+\mu^2\right)\left((p-k')^2+\mu^2\right)} \\ & -\frac{(-1)\left(\not{k}'+im\right)_{ha}\gamma^5_{ab}(-1)\left(\not{k}+im\right)_{be}\gamma^5_{ef}(-1)\left(\not{p}'+im\right)_{fc}\gamma^5_{cd}(-1)\left(\not{p}+im\right)_{dg}\gamma^5_{gh}}{\left((p-p')^2+\mu^2\right)\left((p-k')^2+\mu^2\right)} \\ & \left.+\frac{(-1)\left(\not{p}'+im\right)_{fa}\gamma^5_{ab}(-1)\left(\not{k}+im\right)_{be}\gamma^5_{ef}(-1)\left(\not{k}'+im\right)_{hc}\gamma^5_{cd}(-1)\left(\not{p}+im\right)_{dg}\gamma^5_{gh}}{\left((p-p')^2+\mu^2\right)^2}\right] \end{aligned} \tag{45}$$

$$\begin{aligned} = {} & \frac{g^4}{4}\left[\frac{\text{Tr}\left[\left(\not{k}'+im\right)\gamma^5\left(\not{k}+im\right)\gamma^5\right]\text{Tr}\left[\left(\not{p}'+im\right)\gamma^5\left(\not{p}+im\right)\gamma^5\right]}{\left((p-p')^2+\mu^2\right)^2}\right. \\ & -\frac{\text{Tr}\left[\left(\not{p}'+im\right)\gamma^5\left(\not{k}+im\right)\gamma^5\left(\not{k}'+im\right)\gamma^5\left(\not{p}+im\right)\gamma^5\right]}{\left((p-p')^2+\mu^2\right)\left((p-k')^2+\mu^2\right)} \\ & -\frac{\text{Tr}\left[\left(\not{k}'+im\right)\gamma^5\left(\not{k}+im\right)\gamma^5\left(\not{p}'+im\right)\gamma^5\left(\not{p}+im\right)\gamma^5\right]}{\left((p-p')^2+\mu^2\right)\left((p-k')^2+\mu^2\right)} \\ & \left.+\frac{\text{Tr}\left[\left(\not{p}'+im\right)\gamma^5\left(\not{k}+im\right)\gamma^5\right]\text{Tr}\left[\left(\not{k}'+im\right)\gamma^5\left(\not{p}+im\right)\gamma^5\right]}{\left((p-p')^2+\mu^2\right)^2}\right] \end{aligned} \tag{46}$$

To progress further, we need to make use of the following trace identities:

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma^5\gamma^5\right] = 4\\
&\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^5\right] = \mathrm{Tr}\left[\gamma^\mu\gamma^5\gamma^5\right] = 0\\
&\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^5\gamma^\nu\right] = -4g^{\mu\nu}\\
\\
&\mathrm{Tr}\left[\gamma^5\gamma^5\gamma^5\gamma^\mu\gamma^5\right] = 0\\
&\mathrm{Tr}\left[\gamma^5\gamma^5\gamma^\mu\gamma^5\gamma^\nu\gamma^5\right] = -4g^{\mu\nu}\\
&\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^5\gamma^5\gamma^\nu\gamma^5\right] = 4g^{\mu\nu}\\
&\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^5\gamma^\nu\gamma^5\gamma^\rho\gamma^5\right] = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\right] = 0\\
&\mathrm{Tr}\left[\gamma^\mu\gamma^5\gamma^\nu\gamma^5\gamma^\rho\gamma^5\gamma^\sigma\gamma^5\right] = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)
\end{aligned}
\tag{47}
$$

Using those, the amplitude can be written as:

$$
\begin{aligned}
\frac{1}{2}\sum_s\frac{1}{2}\sum_r\sum_{s',r'}|\mathcal{M}|^2 = \frac{g^4}{4}\Bigg[&\frac{4(-k'_\mu k^\mu - m^2)4(-p'_\mu p^\mu - m^2)}{\left((p-p')^2+\mu^2\right)^2} + \frac{4(-p'_\mu k^\mu - m^2)4(-k'_\mu p'^\mu - m^2)}{\left((p-k')^2+\mu^2\right)^2}\\
&-2\frac{4\left[m^4 + m^2\left(-p'_\mu k'^\mu + k'_\mu k^\mu + k'_\mu p^\mu + k_\mu p'^\mu - k_\mu p^\mu + p'_\mu p^\mu\right) + k'_\mu k^\mu p'_\nu p^\nu - k'p'k_\nu p^\nu + k'_\mu p^\mu k_\nu p'^\nu\right]}{\left((p-p')^2+\mu^2\right)\left((p-k')^2+\mu^2\right)}\Bigg]
\end{aligned}
\tag{48}
$$

To simplify things further it is easier to work in the centre-of-mass frame:

$$
\begin{aligned}
p &= (E, \vec{p} = \sqrt{E^2-m^2}\,\hat{z}) & k &= (E, -\vec{p} = -\sqrt{E^2-m^2}\,\hat{z})\\
p' &= (E, 0, \sqrt{E^2-m^2}\sin\theta, \sqrt{E^2-m^2}\cos\theta) & k' &= (E, 0, -\sqrt{E^2-m^2}\sin\theta, -\sqrt{E^2-m^2}\cos\theta)
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
p'_\mu k'^\mu &= -2E^2 + m^2 & k_\mu p^\mu &= -2E^2 + m^2\,;\\
p'_\mu k^\mu &= -E^2 - (E^2-m^2)\cos\theta & k'_\mu p^\mu &= -E^2 - (E^2-m^2)\cos\theta\,;\\
p'_\mu p^\mu &= -E^2 + (E^2-m^2)\cos\theta & k'_\mu k^\mu &= -E^2 + (E^2-m^2)\cos\theta\,.
\end{aligned}
\tag{50}
$$

We then obtain:

$$
\begin{aligned}
\frac{1}{2}\sum_s\frac{1}{2}\sum_r\sum_{s',r'}|\mathcal{M}|^2 = 4g^4\left(E^2-m^2\right)^2\Bigg[&4\left\{\frac{2(E^2-m^2)\sin^2\theta + \mu^2}{\left(4(E^2-m^2)\sin(\theta/2)+\mu^2\right)\left(4(E^2-m^2)\cos(\theta/2)+\mu^2\right)}\right\}^2\\
&-\frac{\sin^2\theta}{\left(4(E^2-m^2)\sin(\theta/2)+\mu^2\right)\left(4(E^2-m^2)\cos(\theta/2)+\mu^2\right)}\Bigg]\,.
\end{aligned}
\tag{51}
$$

This result can also be written in terms of the Mandelstam variables:

$$
\begin{aligned}
s &= (p+k)^2 = (p'+k')^2 = -4E^2\,;\\
t &= (p-p')^2 = 2\left(E^2-m^2\right)(1-\cos\theta)\,;\\
u &= (p-k')^2 = 2\left(E^2-m^2\right)(1+\cos\theta)\,,
\end{aligned}
$$

to obtain:

$$
\frac{1}{2}\sum_s\frac{1}{2}\sum_r\sum_{s',r'}|\mathcal{M}|^2 = g^4\left[\frac{t^2}{(t+\mu^2)^2} + \frac{u^2}{(u+\mu^2)^2} + \frac{ut}{(t+\mu^2)(u+\mu^2)}\right]
\tag{52}
$$

The differential cross-section is therefore:

$$
\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{g^4}{64\pi^2E_{CM}^2}\left[\frac{t^2}{(t+\mu^2)^2} + \frac{u^2}{(u+\mu^2)^2} + \frac{ut}{(t+\mu^2)(u+\mu^2)}\right]
\tag{53}
$$

To find the amplitude for the process $\bar{\psi}\psi \to \bar{\psi}\psi$, we can make use of the crossing symmetry to recycle the results we obtained for fermions scattering. Let us first consider the first graph, the uncrossed one. To obtain the corresponding uncrossed fermion

- anti-fermion scattering graph, we need to multiply by $(-1)^2 = 1$ for the two fermions interchanged between initial and final states. We also need to make the following momenta substitution: $k \to k$, $k' \to -p$, $p \to -k'$, and $p' \to p'$. To obtain the amplitude for the crossed diagram, we need to multiply the uncrossed graph by -1 (for the interchange of two fermions) and interchange the momenta in the two out-going legs (as was the case for fermions scattering). These substitution change the Mandelstam variables via

$$\begin{aligned} s &= (p+k)^2 \to (-k'+k)^2 = t\,; \\ t &= (p-p')^2 \to (-k'-p')^2 = s\,; \\ u &= (p-k')^2 \to (-k'+p)^2 = u\,. \end{aligned} \tag{54}$$

We therefore obtain the following scattering differential cross-section for the fermion - anti-fermion scattering:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{g^4}{64\pi^2 E_{CM}^2}\left[\frac{s^2}{(s+\mu^2)^2} + \frac{u^2}{(u+\mu^2)^2} + \frac{us}{(s+\mu^2)(u+\mu^2)}\right]. \tag{55}$$