Introduction to String Theory

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Exercise Sheet 1

1 Consider a coordinate change

$$x^\mu \longrightarrow x'^\mu \,. \tag{1.1}$$

Recall that the components of tensors transform in a well-defined way under the coordinate change (1.1). For example, the components of a $(1,1)$ tensor $T^\mu{}_\nu$ transform under (1.1) as

$$T^\mu{}_\nu \longrightarrow T'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}\,T^\rho{}_\sigma \,. \tag{1.2}$$

Similarly, the components of a $(1,1)$ tensor density $\tilde{T}^\mu{}_\nu$ of weight w transforms under (1.1) as

$$\tilde{T}^\mu{}_\nu \longrightarrow \tilde{T}'^\mu{}_\nu = |J|^w \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}\,T^\rho{}_\sigma \,, \tag{1.3}$$

where $J = \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)$ is the Jacobian of the coordinate change (1.1).
(a) Given a $(2,0)$ tensor $S_{\mu\nu}$, show that $(\det S_{\mu\nu})^{1/2}$ is a scalar density of weight 1.
(b) Consider now tensor and tensor density fields, i.e. $T^\mu{}_\nu(x)$, etc. The transformation (1.1) can be expanded infinitesimally as

$$x'^\mu = x^\mu - \epsilon^\mu(x) + \ldots \,. \tag{1.4}$$

Show the following infinitesimal variations for a scalar fields $\Phi(x)$, Lorentzian metric $g_{\mu\nu}(x)$ and its associated density $(-\det g(x))^{1/2}$:

$$\begin{aligned} \delta\Phi &= \epsilon^\mu \partial_\mu \Phi \,, \\ \delta g_{\mu\nu} &= \epsilon^\gamma \partial_\gamma g_{\mu\nu} + (\partial_\mu \epsilon^\gamma) g_{\gamma\nu} + (\partial_\nu \epsilon^\gamma) g_{\mu\gamma} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \,, \\ \delta(-\det g)^{1/2} &= \partial_\gamma(\epsilon^\gamma (-\det g)^{1/2}) \,. \end{aligned} \tag{1.5}$$

The second equality of the second line of (1.5) uses the torsion-free metric-compatible connection ∇_μ.
Hint: A scalar field is defined via $\Phi'(x') = \Phi(x)$ and $\delta\Phi$ in (1.5) is defined as $\delta\Phi(x) \equiv \Phi'(x) - \Phi(x)$ and similarly for the other fields in (1.5).

2 Consider a submanifold H of M defined by the embedding $\mathcal{Y} : H \hookrightarrow M$. A Riemannian metric g on M then induces a pull-back metric $h = \mathcal{Y}^* g$ on H, explicitly given by

$$ds^2 = d\mathcal{Y}^a \, d\mathcal{Y}^b \, g_{ab} \,, \tag{2.1}$$

where $a, b = 1, \ldots, \dim \mathrm{M}$, and $d\mathcal{Y}^a$ denotes the exterior derivative of $\mathcal{Y}^a$ on H. In local coordinates σ^α, $\alpha = 1, \ldots, \dim H$, on H, these are explicitly given by

$$d\mathcal{Y}^a = \frac{\partial \mathcal{Y}^a}{\partial \sigma^\alpha}\, d\sigma^\alpha \,. \tag{2.2}$$

Consider the unit-radius n-sphere $S^n \subset \mathbb{R}^{n+1}$ with the embedding

$$\mathcal{Y} : S^n \hookrightarrow \mathbb{R}^{n+1} \,, \tag{2.3}$$

and where $\mathbb{R}^{n+1}$ is endowed with the flat Riemannian metric $g_{ab} = \delta_{ab}$, such that the embedding satisfies $\mathcal{Y}^a\,\mathcal{Y}^b\,\delta_{ab} = 1$.

(a) Compute the pull-back metric (2.1) for S^2 using spherical coordinates

$$\mathcal{Y}^a = (\cos\phi\,\sin\theta,\, \sin\phi\,\sin\theta,\, \cos\theta)\ , \tag{2.4}$$

with $\phi \in [0,\, 2\pi]$ and $\theta \in [0,\, \pi)$.

(b) Compute the pull-back metric (2.1) for S^n using local coordinates

$$\mathcal{Y}^a = \left(y^i,\, \sqrt{1 - y^i\,\delta_{ij}\,y^j}\right)\ , \tag{2.5}$$

where $i = 1, \ldots, n$.

(c) Compute the pull-back metric (2.1) for $\mathcal{Y} : H^{p,n-p} \hookrightarrow \mathbb{R}^{n+1}$, where $\mathbb{R}^{n+1}$ is endowed with the flat pseudo-Riemannian metric

$$\eta_{ab} = \mathrm{diag}(\underbrace{-1,\, \ldots,\, -1}_{\text{n}-\text{p}},\, \underbrace{+1,\, \ldots,\, +1}_{\text{p}+1})\,, \tag{2.6}$$

and the embedding $\mathcal{Y}^a$ satisfies $\mathcal{Y}^a\,\mathcal{Y}^b\,\eta_{ab} = 1$.

Hint: Use local coordinates on $H^{p,n-p}$ such that

$$\mathcal{Y}^a = \left(y^i,\, \sqrt{1 - y^i\,\eta_{ij}\,y^j}\right)\ , \tag{2.7}$$

where $i = 1, \ldots, n$ and η_{ij} denotes the first $n \times n$ block of η_{ab} in (2.6).

3 Consider the action for a field $\phi(x)$ in a d-dimensional spacetime with Minkowski metric $\eta_{\mu\nu} = (-1,\, +1,\, \ldots,\, +1)$. The action is given in terms of the Lagrangian density by

$$S[\phi] = \int d^dx\, \mathcal{L}(\phi(x), \partial_\mu\phi(x))\,. \tag{3.1}$$

(a) Use integration by parts and assume that you can neglect boundary terms, to show that the action is minimised $\delta S = 0$ for field configurations satisfying the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi}\,. \tag{3.2}$$

(b) Consider the Klein-Gordon action for a massive scalar field ϕ in d dimensions with Minkowski metric

$$S = -\frac{1}{2}\int d^dx\,\left(\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2\right) \tag{3.3}$$

Use the action (3.3) to compute the Euler-Lagrange equations for the scalar field ϕ.

(c) Show that the action (3.3) is invariant under translations and use Noether's Theorem to compute the stress-energy tensor associated with this translation invariance.

(d) Compute the stress-energy tensor in the following alternative way. Consider a position-dependent translation

$$x^\mu \longrightarrow x^\mu + \epsilon^\mu(x)\,. \tag{3.4}$$

The variation of the action (3.3) takes the form

$$\delta S = \int d^dx\, T^{\mu\nu}\,\partial_\mu\epsilon_\nu\,. \tag{3.5}$$

Compute $T^{\mu\nu}$ from (3.5).
(e) Convince yourself that the variation of any translation-invariant action under a position-dependent translation (3.4) must take the form (3.5). Moreover, use (3.5) to argue that $T_{\mu\nu}$ must be conserved when the equations of motion hold.
Hint: Consider what happens when ϵ is constant and what it means for the action to be extremised.
(f) Compute the stress-energy tensor in the following alternative way. Minimally couple the Klein-Gordon action to a dynamical metric $g_{\mu\nu}$ and compute the stress-energy tensor via

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g=\eta} , \tag{3.6}$$

where $|_{g=\eta}$ denotes that we are evaluating with the metric g set to the Minkowski metric η.
(g) Think about the translation-dependent shift (3.4) as a diffeomorphism to show that the stress-energy tensors computed in (3.5) and (3.6) must be the same for any action.

4 Consider the following 1-dimensional action

$$S = \frac{1}{2}\int_{\mathcal{P}} d\tau \left(e^{-1}\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - e\, m^2 \right) , \tag{4.1}$$

where $X^\mu(\tau)$, $e(\tau)$ are dynamical fields, $\dot{X}^\mu = \frac{dX^\mu}{d\tau}$, m is a constant and $\eta_{\mu\nu}$ is the Minkowski metric with $\mu = 0, \ldots, n$.
(a) Given that $X^\mu(\tau)$ are scalar fields, how does e have to transform under reparameterisations

$$\tau \longrightarrow \tilde{\tau}(\tau) , \tag{4.2}$$

such that the action (4.1) is invariant?
(b) Show that the equations of motion for e imply

$$e^2 m^2 + \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} = 0 . \tag{4.3}$$

Plug (4.3) back into the action (4.1) to show that the action is equivalent to

$$S = -m \int_{\mathcal{P}} d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} . \tag{4.4}$$

This is the world-line action of a point-particle in Minkowski space.
(c) Show that the action (4.4) is the same as

$$S = -m \int_{\mathcal{P}} d\tau \sqrt{-\det(X^*\eta)} , \tag{4.5}$$

where $X^*\eta$ is the pull-back of the Minkowski metric η to the worldline of the particle, $\mathcal{P}$, embedded in Minkowski space: $X : \mathcal{P} \hookrightarrow \mathbb{R}^{1,n-1}$.
(d) Compute the equation of motion for the action (4.4). Write the equation of motion in terms of the canonical momentum for X^μ and interpet this equation.
(e) Consider a charged point-particle in Minkowski space coupled to an electromagnetic field

$$S = -m \int_{\mathcal{P}} d\tau \sqrt{-\det(X^*\eta)} + q \int_{\mathcal{P}} d\tau\, X^*A , \tag{4.6}$$

where X^*A denotes the pull-back of the gauge potential A_μ to the worldline of the particle. Convince yourself that the action is explicitly given by

$$S = -m \int_{\mathcal{P}} d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}} + q \int_{\mathcal{P}} d\tau\, A_\mu(X(\tau))\frac{dX^\mu}{d\tau}\,. \tag{4.7}$$

Compute and interpret the equation of motion for the action (4.6).
Hint: Recall the definition of the electromagnetic field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
(f) Consider a neutral point-particle moving in a curved space-time (M, g), i.e.

$$S = -m \int_{\mathcal{P}} d\tau \sqrt{-\det(X^* g)}\,, \tag{4.8}$$

where X^*g is now the pull-back of the metric g to the world-line of the particle, $\mathcal{P}$, embedded in M via $X : \mathcal{P} \hookrightarrow M$. Show that the equation of motion of (4.8) is given by

$$\frac{d^2X^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dX^\nu}{ds}\frac{dX^\rho}{ds} = 0\,, \tag{4.9}$$

where

$$ds = \sqrt{-\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}}\, d\tau\,, \tag{4.10}$$

and $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols given explicitly by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right)\,. \tag{4.11}$$

Interpret this equation of motion.
(g) The action (4.8) can be generalised to a 1-dimensional string. This leads to the Nambu-Goto action:

$$S = -T \int_\Sigma d^2\sigma \sqrt{-\det(X^* g)}\,, \tag{4.12}$$

where T is the tension of the string, X^*g is now the pull-back of g to Σ, the worldsheet of the string, and $\sigma^\alpha = (\tau, \sigma)$ denote local coordinates on the worldsheet of the string. Write out (4.12) explicitly.