Introduction to String Theory

Humboldt-Universität zu Berlin
Dr. Emanuel Malek

Exercise Sheet 10

1 Consider a point particle whose worldline is an S^1, i.e. $\tau \in (0,1]$ with periodicity condition

$$X(0) = X(1)\,, \qquad e(0) = e(1)\,, \tag{1.1}$$

where $e(\tau)$ is the worldline einbein, appearing in the point particle worldline action as

$$S = \frac{1}{2}\int_{S^1} d\tau \left(e^{-1}\dot{X}^2 - e\,m^2\right)\,. \tag{1.2}$$

(a) Consider a reparameterisation $\tau \longrightarrow \tilde{\tau}(\tau)$ such that

$$\tilde{e}(\tilde{\tau}) = 1\,. \tag{1.3}$$

What is the range of the new S^1 coordinate, $\tilde{\tau}$?

(b) Now consider a reparameterisation $\tau \longrightarrow \tilde{\tau}(\tau)$ such that

$$\tilde{e} = \int_0^1 e(\tau)d\tau\,. \tag{1.4}$$

What is now the range of the new S^1 coordinate, $\tilde{\tau}$?

(c) What is the moduli space of the einbein e on S^1?

(d) After imposing the condition of $\tilde{e} = \text{const.}$, is there any remnant gauge symmetry left?

2 Consider the Virasoro algebra

$$[L_m,\,L_n] = (m-n)L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0}\,. \tag{2.1}$$

(a) Show that L_{-1}, L_0, L_1 form a subalgebra of the Virasoro algebra. What algebra do they form and how does it depend on the central charge?

(b) Consider the conformal Killing vector fields $l_n = -z^{n+1}\partial_z$ and show that they generate the Witt algebra

$$[l_m,\,l_n] = (m-n)\,l_{m+n}\,. \tag{2.2}$$

Which l_n are non-singular at $z = 0$?

(c) Which conformal Killing vector fields l_n are non-singular at $z = \infty$?

(d) What are the infinitesimal diffeomorphisms generated by l_{-1}, l_0 and l_1? Integrate each of these up to a finite coordinate transformation.

(e) Argue that the general finite coordinate transformation generated by l_{-1}, l_0 and l_1 is

$$z \longrightarrow \frac{a\,z+b}{c\,z+d}\,. \tag{2.3}$$

These are called *fractional linear transformations.* Explain why there are four parameters a, b, c and d even though there are only three tranformations corresponding to l_{-1}, l_0 and l_1?

(f) By considering the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})\,, \tag{2.4}$$

associated to the fractional linear transformation

$$z \longrightarrow \frac{a\,z+b}{c\,z+d}\,, \tag{2.5}$$

argue that the group of fractional linear transformations is isomorphic to $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

3 Optional: Consider the map

$$z \longrightarrow z' = \frac{(b-c)(z-a)}{(b-a)(z-c)}\,, \tag{3.1}$$

where $a \neq b$, $b \neq c$ and $a \neq c$.

(a) Find the $\mathrm{SL}(2,\mathbb{C})$ matrix corresponding to this fractional linear transformation. Show that this $\mathrm{SL}(2,\mathbb{C})$ transformation maps any distinct points on S^2 to any other 3 distinct points.

(b) Show that the cross-ratio

$$\langle z_1,\, z_2,\, z_3,\, z_4\rangle \equiv \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}\,, \tag{3.2}$$

is $\mathrm{SL}(2,\mathbb{C})$-invariant. Use this to show that

$$\langle z,\, a,\, b,\, c\rangle = z'\,, \tag{3.3}$$

with z' given in (3.1).
Hint: What point does the fractional linear transformation

$$z \rightarrow \frac{a\,z+b}{c\,z+d}\,, \tag{3.4}$$

map $z=\infty$ to?