Introduction to String Theory

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Exercise Sheet 11

1 Explain why the scattering amplitude with b and c ghost insertions

$$\begin{aligned} A^{(n)}(\Lambda_i, p_i) = &\sum_{\text{topologies}} g_s^{-\chi} \int \prod_{I=1}^{\mu} dm^I \int DX\, Db\, Dc\, e^{-S_{\text{Poly}}-S_{\text{ghost}}} \\ &\times \prod_{I=1}^{\mu} \frac{1}{4\pi}(b, \partial_I \hat{g}) \prod_{i=1}^{\kappa} c(z_i)\bar{c}(\bar{z}_i) V_{\Lambda_i}(p_i, z_i, \bar{z}_i) \prod_{i=\kappa+1}^{n} \int d^2 z_i\, V_{\Lambda_i}(p_i, z_i, \bar{z}_i)\,, \end{aligned} \tag{1.1}$$

is BRST invariant.
Hint: Recall that the BRST variation of the b ghost field is the stress-energy tensor and assume that the moduli space is compact without boundary.

2 Consider the correlation function of $n+3$ c-ghosts and n b-ghosts on S^2, for some $n \in \mathbb{Z}$, i.e.

$$\langle \prod_{i=1}^{n+3} c(z_i) \prod_{i'=1}^{n} b(\tilde{z}_{i'}) \rangle_{S^2\,\text{ghost}}\,. \tag{2.1}$$

(a) What are the zeros and poles of the correlation function (2.1)? Write down a meromorphic function that has the right zeros and poles.

(b) How does (2.1) behave as one of the $z_i \longrightarrow \infty$? How does (2.1) behave as one of the $\tilde{z}_{i'} \longrightarrow \infty$? Use this to determine (2.1) up to a constant.

3 The 4-point tachyon tree-level amplitude is given by

$$\begin{aligned} A^{(4)}_{\text{tree}}(T, p_1, p_2, p_3, p_4) \sim\, &g_s^2\, |z_{12}|^{2+\alpha' p_1\cdot p_2} |z_{23}|^{2+\alpha' p_2\cdot p_3} |z_{13}|^{2+\alpha' p_1\cdot p_3}\, \delta(\sum_i p_i) \\ &\times \int d^2 z\, |z_1 - z|^{\alpha' p_1\cdot p_4} |z_2 - z|^{\alpha' p_2\cdot p_4} |z_3 - z|^{\alpha' p_3\cdot p_4}\,. \end{aligned} \tag{3.1}$$

(a) We can use $\mathrm{PSL}(2,\mathbb{C})$ to fix $z_1 = 0$, $z_2 = 1$, $z_3 = \lambda \to \infty$. Show that upon doing this, the terms involving z_3 cancel and the 4-point amplitude (3.1) becomes

$$A^{(4)}_{\text{tree}}(T, p_1, p_2, p_3, p_4) \sim g_s^2\, \delta(\sum_i p_i) \int d^2 z\, |z|^{\alpha' p_1\cdot p_4} |1 - z|^{\alpha' p_2\cdot p_4}\,. \tag{3.2}$$

(b) The Γ function is defined as

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1}\, e^{-t}\,, \qquad z \in \mathbb{C}\,. \tag{3.3}$$

Use a saddle-point approximation to prove Stirling's formula

$$\lim_{z\to\infty} \Gamma(z) \sim \exp(z \ln z)\,. \tag{3.4}$$

(c) Use the fact that the 4-point amplitude can be written as

$$A^{(4)}_{\text{tree}}(T, p_1, p_2, p_3, p_4) \sim g_s^2\, \delta(\sum_i p_i) \frac{2\pi\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)}\,, \tag{3.5}$$

and (3.4) to show that in the limit $s \longrightarrow \infty$, $t \longrightarrow \infty$, with s/t fixed, we get

$$A^{(4)}_{\text{tree}}(T, p_1, p_2, p_3, p_4) \sim \exp\left(-\frac{\alpha'}{2}(s \ln s + t \ln t + u \ln u)\right)\,. \tag{3.6}$$

4 You will now use a holomorphicity argument as in question (2) to compute a correlation function of the type

$$\langle \partial X^\mu(z) \prod_{i=1}^{n} : e^{ip_i \cdot X(z_i, \bar{z}_i)} : \rangle_{S^2, \text{Poly}}\,. \tag{4.1}$$

(a) Use the $\partial X(z) : e^{ip_i \cdot X(z', \bar{z}')} :$ OPE to determine (4.1) in terms of the Tachyon n-point correlator

$$A^{(n)}_{T,\text{tree}}(p_i, z_i, \bar{z}_i) = \langle \prod_{i=1}^{n} : e^{ip_i \cdot X(z_i, \bar{z}_i)} : \rangle_{S^2, \text{Poly}}\,, \tag{4.2}$$

and up to the addition of functions holomorphic in z.

(b) Use the fact that (4.1) should be well-defined as $z \longrightarrow \infty$ to argue that

$$\sum_{i=1}^{n} p_i^\mu = 0\,, \tag{4.3}$$

and

$$\langle \partial X^\mu(z) \prod_{i=1}^{n} : e^{ip_i \cdot X(z_i, \bar{z}_i)} : \rangle_{S^2, \text{Poly}} = -\frac{i\,\alpha'}{2} A^{(n)}_{T,\text{tree}}(p_i, z_i, \bar{z}_i) \sum_{i=1}^{n} \frac{p_i^\mu}{z - z_i}\,. \tag{4.4}$$

Note: The actual correlation function that we need to compute an amplitude involving a higher-level particle is

$$\langle : \partial X^\mu(z) e^{ip \cdot X(z, \bar{z})} : \prod_{i=1}^{n} : e^{ip_i \cdot X(z_i, \bar{z}_i)} : \rangle_{S^2, \text{Poly}} = -\frac{i\,\alpha'}{2} A^{(n)}_{T,\text{tree}}(p_i, z_i, \bar{z}_i) \sum_{i=1}^{n} \frac{p_i^\mu}{z - z_i}\,. \tag{4.5}$$

This can be deduced from the result (4.4) by arguing that the normal ordering removes the singularity as $z \longrightarrow z_i$.

5 Optional: Consider the gauge-fixed path integral with vertex operator insertions and metric moduli. We want to fix the position of κ vertex operators at $\hat{z}_i$, $i = 1, \ldots, \kappa$. We therefore define the Faddeev-Popov determinant with moduli and fixed positions of κ vertex operators as

$$1 = \Delta_{FP}(\hat{g}, \hat{z}_i) \prod_{I=1}^{\mu} \int dm^I \int D\zeta\, \delta(\hat{g} - \hat{g}(m)^\zeta) \prod_{i=1}^{\kappa} \delta(\hat{z}_i - \hat{z}_i^\zeta)\,. \tag{5.1}$$

(a) How does the moduli-dependent metric $\hat{g}(m)$ change under a diffeomorphism + Weyl transformation? I.e. what is $\hat{g}(m)^\zeta$?

(b) How do the positions of the insertions $\hat{z}_i$ change under a diffeomorphism + Weyl transformations? I.e. what is $\hat{z}_i^\zeta$?

(c) Give a path-integral representation of Δ_{FP}^{-1}.

(d) Using the usual trick of replacing commuting variables with Grassman fields to invert the determinant, argue that the Faddeev-Popov determinant $\Delta_{FP}(\hat{g}, \hat{z}_i)$ is given by

$$\Delta_{FP}(\hat{g}, \hat{z}_i) = \int Db\, Dc\, D^\mu\gamma\, D^\kappa\eta\, \exp\left(-\frac{1}{4\pi}(b, P\cdot c - \gamma^I \partial_{t^I}\hat{g}) + \sum_{i=1}^{\kappa} \eta_{\alpha\, i} c^\alpha(\hat{z}_i)\right) . \tag{5.2}$$

(e) Perform the Grassmann path integral over γ and η to obtain the final result

$$\Delta_{FP}(\hat{g}, \hat{z}_i) = \int Db\, Dc\, e^{-S_{\text{ghost}}} \prod_{I=1}^{\mu} \frac{1}{4\pi}(b, \partial_I \hat{g}) \prod_{i=1}^{\kappa} c^\alpha(\hat{z}_i) . \tag{5.3}$$

(f) Show that the n-point amplitudes, with $n \geq \kappa$ reduces to the form given in the lectures, i.e. with κ unintegrated vertex operators (involving c-ghost insertions), $n - \kappa$ integrated vertex operators and b-ghost insertions for each modulus.

6 Optional: Consider

$$\Gamma(x)\Gamma(y) = \int_0^\infty du \int_0^\infty dv\, e^{-u} e^{-v} u^{x-1} v^{y-1} . \tag{6.1}$$

By changing to coordinates $u = a^2$ and $v = b^2$, show that the Euler Beta function, defined as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} , \tag{6.2}$$

is given by

$$B(x, y) = \int_0^1 dt\, t^{x-1}\, (1-t)^{y-1} . \tag{6.3}$$

7 Optional: The Γ function is defined as

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1}\, e^{-t} , \qquad z \in \mathbb{C} . \tag{7.1}$$

(a) Show that

$$|z|^{2a-2} = \frac{1}{\Gamma(1-a)} \int_0^\infty dt\, t^{-a}\, e^{-|z|^2 t} . \tag{7.2}$$

(b) Use the result (7.2) to show that

$$\int d^2z\, |z|^{2a-2} |1-z|^{2b-2} = \frac{2\pi\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)} \int_0^1 d\beta (1-\beta)^{a-1} \beta^{b-1} , \tag{7.3}$$

with $a + b + c = 1$.

(c) Using (6.3), show that

$$\int d^2z\, |z|^{2a-2} |1-z|^{2b-2} = \frac{2\pi\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)} , \tag{7.4}$$

with $a + b + c = 1$.