Introduction to String Theory

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Exercise Sheet 12

1 In this exercise you will show that the torus partition function of the string

$$A^{0)} \sim \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\tau)^2} \left(\sqrt{\mathrm{Im}\tau}\, |\eta(q)|^2\right)^{-24} , \tag{1.1}$$

is modular invariant.

(a) Show that the measure

$$\frac{d^2\tau}{(\mathrm{Im}\tau)^2} , \tag{1.2}$$

is invariant under fractional linear transformations

$$\tau \to \frac{a\tau + b}{c\tau + d} . \tag{1.3}$$

(b) The Dedekind η-function satisfies

$$\eta(\tau + 1) = e^{2\pi i/24}\eta(\tau) , \qquad \eta(-\frac{1}{\tau}) = \sqrt{-i\,\tau}\, \eta(\tau) . \tag{1.4}$$

Use this to show that the combination

$$\mathrm{Im}\tau\, |\eta(\tau)|^4 \tag{1.5}$$

is invariant under S and T transformations

$$T : \tau \to \tau + 1 , \qquad S : \tau \to -\frac{1}{\tau} . \tag{1.6}$$

2 The action of a string coupled to the background fields G and B is given by

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{g} \left(g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) + i\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) \right) . \tag{2.1}$$

Show that the action (2.1) is invariant under spacetime diffeomorphisms and B-field gauge transformations

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu . \tag{2.2}$$

3 **(a)** Show that the combination

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} , \tag{3.1}$$

is invariant under the B-field gauge transformations

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu . \tag{3.2}$$

(b) Let the string worldsheet, Σ, be the boundary of a 3-manifold, M_3, i.e. $\Sigma = \partial M_3$. Show that the string action (2.1) can be written as

$$\begin{aligned} S = &\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) \\ &+ \frac{i}{12\pi\alpha'}\int_{M_3} d^3\hat{\sigma}\,\eta^{\hat{\alpha}\hat{\beta}\hat{\gamma}}\partial_{\hat{\alpha}}X^\mu \partial_{\hat{\beta}}X^\nu \partial_{\hat{\gamma}}X^\rho H_{\mu\nu\rho}(X)\,, \end{aligned} \tag{3.3}$$

where $\hat{\sigma}^{\hat{\alpha}}$ are local coordinates on M_3 and $\eta^{\hat{\alpha}\hat{\beta}\hat{\gamma}}$ is the totally antisymmetric 3-dimensional Levi-Civita tensor *density* with $\eta^{123} = 1$.

4 Consider the action

$$S = \frac{1}{2\kappa_0^2}\int d^{26}X\sqrt{-G}e^{-2\Phi}\left(R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\right)\,. \tag{4.1}$$

Show that its equations of motion for G, B and Φ are given by

$$\begin{aligned} 0 &= R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\rho\lambda}H_\nu{}^{\rho\lambda}\,, \\ 0 &= -\frac{1}{2}\nabla^\rho H_{\mu\nu\rho} + H_{\mu\nu\rho}\nabla^\rho\Phi\,, \\ 0 &= -\frac{1}{2}\nabla^2\Phi + \nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}\,. \end{aligned} \tag{4.2}$$