Introduction to String Theory

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Exercise Sheet 13

1 Consider the low-energy effective action in string frame in D spacetime dimensions, given by

$$S = \frac{1}{2\kappa_0^2} \int d^D X \sqrt{-G} e^{-2\Phi} \left(R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial_\mu \Phi \partial^\mu \Phi \right) . \tag{1.1}$$

Show that, when written in terms of the Einstein frame metric

$$\tilde{G}_{\mu\nu}(X) = e^{-4\tilde{\Phi}/(D-2)} G_{\mu\nu}(X) , \tag{1.2}$$

with $\Phi(X) = \Phi_0 + \tilde{\Phi}(X)$ and Φ_0 constant, the action (1.1) becomes

$$S = \frac{1}{2\kappa^2} \int d^D X \sqrt{-\tilde{G}} \left(\tilde{R} - \frac{1}{12} e^{-8\tilde{\Phi}/(D-2)} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{4}{D-2} \partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} \right) , \tag{1.3}$$

where $\kappa^2 = \kappa_0^2 e^{2\phi_0}$.

2 The string frame metric produced by N infinite static strings lying in the $(X^0, X^1) \equiv (t, x)$ direction is

$$ds^2 = f(r)^{-1} (-dt^2 + dx^2) + d\vec{X} \cdot d\vec{X} , \tag{2.1}$$

where $\vec{X} = \left(X^2, \ldots, X^{25} \right)$ labels the space transverse to the string and

$$f(r) = 1 + \frac{g_s^2 N l_s^{22}}{r^{22}} , \tag{2.2}$$

with $r^2 = \vec{X} \cdot \vec{X}$. Consider one further infinite probe string in this background, lying parallel to the others.

(a) Write down the Nambu-Goto action describing the motion of this string. Show that in static gauge $t = R\tau$ and $x = R\sigma$, the low-energy excitations of the string are governed by the effective action

$$S \sim T \int dt\, dx \left[-f(r)^{-1} + \frac{1}{2} \left(\frac{d\vec{X}}{dt} \cdot \frac{d\vec{X}}{dt} - \frac{d\vec{X}}{dx} \cdot \frac{d\vec{X}}{dx} \right) + \ldots \right] . \tag{2.3}$$

Interpret this result.

(b) Now include the coupling of the probe string to the background B-field, which is given by

$$B_{01} = f(r)^{-1} - 1 . \tag{2.4}$$

Show that the probe string, suitably oriented and lying parallel to the initial strings, feels no static force.

3 Consider an open string whose endpoints are constrained to lie on a Dp-brane, with the closed-string background field $B_{\mu\nu}$ and the open string background field on the D-brane A_a turned on, where X^a label the parallel directions and X^I the normal directions to the Dp-brane. These background fields couple to the open string via the terms

$$S_B + S_A = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{g} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \int_{\partial\Sigma} d\tau A_a \dot{X}^a \,. \tag{3.1}$$

(a) Show that (3.1) is invariant under the A_a gauge transformations

$$\delta A_a = \partial_a \Lambda \,. \tag{3.2}$$

(b) Show that (3.1) is only invariant under the B-field gauge transformations

$$\delta B_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu \,, \tag{3.3}$$

if A_a transforms as

$$\delta A_a = -\frac{1}{2\pi\alpha'} C_a \,. \tag{3.4}$$

(c) Show that the gauge-invariant field strength of A_a on the D-brane worldvolume is given by

$$\mathcal{F}_{ab} = B_{ab} + 2\pi\alpha' F_{ab} \,, \tag{3.5}$$

with

$$F_{ab} = \partial_a A_b - \partial_b A_a \,. \tag{3.6}$$

(d) Show that the Neumann boundary condition for the string must be replaced by

$$\partial_\sigma X^a + \mathcal{F}^{ab} \partial_\tau X_b = 0 \,. \tag{3.7}$$

4 Optional: The Einstein frame metric produced by N D3-branes in superstring theory along the $(X^0, X^1, X^2, X^3) \equiv (t, y^1, y^2, y^p)$ directions, analogous to (2.1) for the string, is given by

$$ds^2 = H^{-1/2}\left(-dt^2 + d\vec{y} \cdot d\vec{y}\right) + H^{1/2} d\vec{x} \cdot d\vec{x} \,, \tag{4.1}$$

with $\vec{x} = \left(X^4, \ldots, X^9\right)$, $r^2 = \vec{x} \cdot \vec{x}$ and

$$H(r) = 1 + \frac{g_{YM}^2 N \alpha'^2}{r^4} \,. \tag{4.2}$$

Here g_{YM} is a constant defined in terms of the dilaton of the solution, given by

$$e^\Phi = g_s = \frac{g_{YM}^2}{4\pi} \,. \tag{4.3}$$

There is also a five-form field strength, but which is not important for the purpose of this question.

(a) Show that in the limit $r \to 0$, the metric becomes the metric on $\mathrm{AdS}_5 \times S^5$ with S^5 radius and AdS_5 radius both given by

$$L = \left(g_{YM}^2 N \alpha'^2\right)^{1/4} \,. \tag{4.4}$$

(b) The solution (4.1) depends on two dimensionless free parameters N and g_{YM}. For what values of N and g_{YM} we can trust the solution (4.1) as a solution of the low-energy effective action of string theory?