Introduction to String Theory

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Exercise Sheet 2

1 Consider the Polyakov-style action for a massless relativistic point-particle

$$S = \frac{1}{2} \int_{\mathcal{P}} d\tau \, e^{-1} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} \,. \tag{1.1}$$

(a) Compute the equations of motion for this action.
(b) Use reparameterisation invariance to gauge-fix $e(\tau)$.
(c) Evaluate the gauge-fixed equations of motion and interpret them.

2 As we will soon see, string theory contains a set of higher-dimensional objects, called *branes*. A $(p+1)$-dimensional brane, is called a p-brane. The dynamics of these objects in Minkowski space is determined by the Dirac action

$$S = -T \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{-\det X^*\eta} \,, \tag{2.1}$$

where Σ_{p+1} denotes the worldvolume of the p-brane, $X : \Sigma_{p+1} \hookrightarrow \mathbb{R}^{1,D-1}$ defines its embedding, σ^α, $\alpha = 0, \ldots, p$ are local coordinates on Σ_{p+1} and η is the Minkowski metric on $\mathbb{R}^{1,D-1}$. As usual, $X^*\eta$ is the pull-back metric on Σ_{p+1} defined as

$$(X^*\eta)_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \,. \tag{2.2}$$

Show that the Dirac action (2.1) is equivalent to the Polyakov-style action

$$S = -\frac{T}{2} \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{-\det g} \left(g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - (p-1) \right) , \tag{2.3}$$

where $g_{\alpha\beta}$ is a dyanmical worldvolume metric.

3 The Polyakov action for the string

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{-\det g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \,, \tag{3.1}$$

is invariant under global Poincaré transformations

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + C^\mu \,. \tag{3.2}$$

(a) Show that the Noether current for the global translation, generated by C^μ in (3.2), is given by

$$P^\alpha_\mu = -T \sqrt{-\det g} \, g^{\alpha\beta} \partial_\beta X_\mu \,, \tag{3.3}$$

and that this is conserved

$$\nabla_\alpha P^\alpha_\mu = 0 \,, \tag{3.4}$$

where ∇ is the worldsheet connection compatible with the worldsheet metric $g_{\alpha\beta}$.
(b) Show that the Noether current for the global Lorentz transformations, generated by $\Lambda^{\mu}{}_{\nu}$ in (3.2), is given by

$$J^{\alpha}_{\mu\nu} = X_{\mu} P^{\alpha}_{\nu} - X_{\nu} P^{\alpha}_{\mu}\,, \tag{3.5}$$

and that this is conserved

$$\nabla_{\alpha} J^{\alpha}_{\mu\nu} = 0\,. \tag{3.6}$$

(c) Show that

$$\begin{aligned} \nabla_{\alpha} P^{\alpha}_{\mu} &= \partial_{\alpha} P^{\alpha}_{\mu} = 0\,, \\ \nabla_{\alpha} J^{\alpha}_{\mu\nu} &= \partial_{\alpha} J^{\alpha}_{\mu\nu} = 0\,, \end{aligned} \tag{3.7}$$

Hint: P^{α}_{μ} and $J^{\alpha}_{\mu\nu}$ are worldsheet vector densities of weight 1.

4 Consider an action $S[\phi, g]$ that is Weyl invariant, i.e. invariant under

$$g_{\alpha\beta} \longrightarrow \Omega^2(x) g_{\alpha\beta}\,. \tag{4.1}$$

Show that the stress-energy tensor must be traceless:

$$T^{\mu}{}_{\mu} = 0\,. \tag{4.2}$$

Hint: Use the definition of the stress-energy tensor as

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}\,. \tag{4.3}$$

5 Consider a modified Polyakov action, where we add a cosmological constant term, i.e.

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{-\det g}\, g^{\alpha\beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \eta_{\mu\nu} + \lambda \int_{\Sigma} d^2\sigma \sqrt{-\det g}\,. \tag{5.1}$$

Show that consistency of the equations of motion for $g_{\alpha\beta}$ require $\lambda = 0$.

6 A conformal Killing vector field, k, of a metric $g_{\mu\nu}$ satisfies

$$L_k g_{\mu\nu} = \nabla_{\mu} k_{\nu} + \nabla_{\nu} k_{\mu} = f(k) g_{\mu\nu}\,. \tag{6.1}$$

(a) Show that in D dimensions

$$f(k) = \frac{2}{D} \nabla_{\mu} k^{\mu}\,. \tag{6.2}$$

(b) Consider the conformal Killing vector fields of Minkowski space in $D \neq 2$ dimensions. First, use the defining equation for conformal Killing vector fields for Minkowski space,

$$\partial_{\mu} k_{\nu} + \partial_{\nu} k_{\mu} = f\, \eta_{\mu\nu}\,, \tag{6.3}$$

to show that

$$(2 - D)\, \partial_{\mu} f = 2 \partial^{\nu} \partial_{\nu} k_{\mu}\,. \tag{6.4}$$

By differentiating (6.4) again, show that

$$\partial_{\mu} \partial_{\nu} f = 0\,, \tag{6.5}$$

and hence a general conformal Killing vector field is given by

$$k_\mu = A_\mu + \Lambda_{\mu\nu}\, x^\nu + B x_\mu + 2\, x_\mu C_\nu x^\nu - C_\mu x_\nu x^\nu \,, \tag{6.6}$$

where A_μ, B, C_μ and $\Lambda_{\mu\nu}$ are constants, with $\Lambda_{(\mu\nu)} = 0$.

(c) Now consider conformal Killing vector fields of 2-dimensional Minkowski space. Show that they satisfy the free wave equation, i.e.

$$\partial_\nu \partial^\nu k_\mu = 0 \,. \tag{6.7}$$

Notice that unlike in $D \neq 2$ dimensions, there are infinitely many conformal Killing vector fields of 2-dimensional Minkowski space.