Introduction to String Theory

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Exercise Sheet 3

1 Consider the Polyakov action

$$S_P = -\frac{T}{2}\int_\Sigma d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\,, \tag{1.1}$$

in flat gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$.
(a) Compute the canonical momenta

$$\Pi_\mu = \frac{\delta S}{\delta \dot{X}^\mu}\,. \tag{1.2}$$

to the scalar fields X^μ.
(b) The Poisson bracket of two function(al)s, F, G of X^μ and Π_μ is defined as

$$\{F,\, G\} \equiv \int_0^{2\pi} d\tilde{\sigma}\left(\frac{\partial F}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial G}{\partial \Pi_\mu(\tau,\tilde{\sigma})} - \frac{\partial G}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial F}{\partial \Pi_\mu(\tau,\tilde{\sigma})}\right)\,. \tag{1.3}$$

Show that this leads to the canonical equal-time Poisson brackets

$$\begin{aligned}\left\{X^\mu(\tau,\sigma),\, \Pi^\nu(\tau,\sigma')\right\} &= \eta^{\mu\nu}\delta(\sigma-\sigma')\,,\\ \left\{X^\mu(\tau,\sigma),\, X^\nu(\tau,\sigma')\right\} &= \left\{\Pi^\mu(\tau,\sigma),\, \Pi^\nu(\tau,\sigma')\right\} = 0\,.\end{aligned} \tag{1.4}$$

(c) Recall the canonical charges for conformal transformations are defined as

$$\begin{aligned}L_{\epsilon^+} &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\,\epsilon^+\,T_{++}\,,\\ L_{\epsilon^-} &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\,\epsilon^-\,T_{--}\,,\end{aligned} \tag{1.5}$$

with $T_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}$.
Show that these generate conformal transformations via the Poisson bracket (1.3)

$$\begin{aligned}\{L_{\epsilon^+},\, X^\mu(\tau,\sigma)\} &= -\epsilon^+\partial_+X^\mu(\tau,\sigma)\,,\\ \{L_{\epsilon^+},\, X^\mu(\tau,\sigma)\} &= -\epsilon^-\partial_-X^\mu(\tau,\sigma)\,.\end{aligned} \tag{1.6}$$

(d) Show that the Poisson bracket of the Virasoro generators

$$\begin{aligned}\tilde{L}_n &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\,T_{++}\,e^{in\sigma^+}\,,\\ L_n &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\,T_{--}\,e^{in\sigma^-}\,,\end{aligned} \tag{1.7}$$

is given by

$$\begin{aligned}\left\{L_m,\, L_n\right\} &= -i(m-n)L_{m+n}\,,\\ \left\{\tilde{L}_m,\, \tilde{L}_n\right\} &= -i(m-n)\tilde{L}_{m+n}\,,\\ \left\{L_m,\, \tilde{L}_n\right\} &= 0\,.\end{aligned} \tag{1.8}$$

This is called the Witt algebra.
(e) Show that the differential operators

$$T_n = -e^{in\sigma^-}\partial_- \,, \qquad \tilde{T}_n = -e^{in\sigma^+}\partial_+ \,, \tag{1.9}$$

satisfy the algebra

$$[T_m,\, T_n] = i(m-n)T_{m+n} \,, \tag{1.10}$$

via the Lie bracket.
We can now write a nice relationship between the Noether charges of conformal transformations and the conformal transformations they generate. Let

$$Q_{T_m} \equiv L_m \,, \tag{1.11}$$

denote the Noether charges associated to the tranformations T_m. Then

$$\{Q_{T_m},\, Q_{T_n}\} = -Q_{[T_m,\, T_n]} \,. \tag{1.12}$$

(f) Now show that the above holds in general, i.e. consider some symmetry under which the fields ϕ transform as $\delta_\epsilon\phi$. Write the associated Noether charges as Q_ϵ, such that

$$\{Q_\epsilon,\, \phi\} = \delta_\epsilon \phi \,. \tag{1.13}$$

Show that

$$\{\{Q_{\epsilon_1},\, Q_{\epsilon_2}\}\,, \phi\} = -(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})\phi \,. \tag{1.14}$$

This is equivalent to saying

$$\{Q_{\epsilon_1},\, Q_{\epsilon_2}\} = -Q_{[\epsilon_1,\, \epsilon_2]} \,. \tag{1.15}$$

Hint: The Poisson bracket satisfies the Jacobi identity

$$\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0 \,. \tag{1.16}$$

2 Consider the mode expansion for the string

$$\begin{aligned} X^\mu_L(\sigma^+) &= \frac{1}{2}\left(x^\mu + c^\mu\right) + \frac{1}{2}\alpha' p^\mu\, \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}^\mu_n\, e^{-in\sigma^+} \,, \\ X^\mu_R(\sigma^-) &= \frac{1}{2}\left(x^\mu - c^\mu\right) + \frac{1}{2}\alpha' p^\mu\, \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\, e^{-in\sigma^-} \,. \end{aligned} \tag{2.1}$$

(a) Show that the center of mass position

$$x^\mu_0 = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, X^\mu \,, \tag{2.2}$$

is, at $\tau = 0$, given by

$$x^\mu_0 = x^\mu \,. \tag{2.3}$$

(b) Using your results from question 3(b) of Exercise Sheet 2 in flat gauge, show that the conserved charge associated to spacetime translations,

$$P_\mu \equiv \int_0^{2\pi} d\sigma\, P^\tau_\mu \,, \tag{2.4}$$

is given by

$$P_\mu = p_\mu \,. \tag{2.5}$$

(c) Using your results from question 3(b) of Exercise Sheet 2 in flat gauge, show that the conserved charge associated to spacetime Lorentz transformations,

$$J_{\mu\nu} \equiv \int_0^{2\pi} d\sigma \, J^\tau_{\mu\nu} \,, \tag{2.6}$$

is given by

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} \,, \tag{2.7}$$

where

$$\begin{aligned} l^{\mu\nu} &= x^\mu \, p^\nu - x^\nu \, p^\mu \,, \\ E^{\mu\nu} &= -i \sum_{n>0} \frac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n \right) \,, \end{aligned} \tag{2.8}$$

and similarly for $\tilde{E}^{\mu\nu}$. Interpret $l^{\mu\nu}$ and $E^{\mu\nu}$.

3 In the quantum theory, the Virasoro generators generate a central extension of the Witt algebra. This takes the form

$$[L_m, \, L_n] = (m-n) L_{m+n} + C(n) \delta_{m+n,0} \,, \tag{3.1}$$

where $C(n)$ is some real function that you will now determine. $C(n)$ is called the central extension of the algebra. Note: we wrote commutators rather than Poisson brackets in (3.1) to emphasise that this will occur in the quantum theory.

(a) Show that skew-symmetry of the commutator implies

$$C(n) = -C(-n) \,. \tag{3.2}$$

(b) By considering the Jacobi identity, show that the function $C(n)$ must take the form

$$C(n) = c_3 \, n^3 + c_1 \, n \,, \tag{3.3}$$

for some constants c_1, c_3.

(c) Compute

$$\langle 0, p | [L_1, L_{-1}] | 0, p \rangle \quad \text{and} \quad \langle 0, p | [L_2, L_{-2}] | 0, p \rangle \,, \tag{3.4}$$

to show that the correct commutation relations are given by the *Virasoro algebra*:

$$[L_m, \, L_n] = (m-n) L_{m+n} + \frac{D}{12} (m^3 - m) \delta_{m+n,0} \,. \tag{3.5}$$