Introduction to String Theory

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Exercise Sheet 4

1 Using the results from Exercise Sheet 3, Question 1, show that $H = L_0 + \tilde{L}_0$ generates τ-translations and $L_0 - \tilde{L}_0$ generates σ-translations. Use this to argue that in canonical quantisation, the Virasoro conditions must involve the same normal-ordering constants, i.e. if we impose

$$(L_0 - a)\,|\mathrm{phys}\rangle = \left(\tilde{L}_0 - \tilde{a}\right)|\mathrm{phys}\rangle = 0\,, \tag{1.1}$$

we should also have $a = \tilde{a}$.

2 Here you will evaluate the string Hamiltonian in light-cone gauge.

(a) Using light-cone gauge,

$$X^+ = x^+ + \alpha' p^+ \tau\,, \tag{2.1}$$

show that the string action

$$S = \int d\tau\, L = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\, \partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu}\,, \tag{2.2}$$

reduces to

$$S = -\int d\tau\, p^+ \dot{q}^- + \frac{1}{4\pi\alpha'}\int d^2\sigma \left(\dot{X}^i\dot{X}^i - X'^i X'^i\right)\,, \tag{2.3}$$

where

$$q^- = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, X^-\,. \tag{2.4}$$

(b) Compute the canonical momenta

$$p_- = \frac{\delta S}{\delta \dot{q}^-}\,, \qquad \Pi_i = \frac{\delta S}{\delta \dot{X}^i}\,, \tag{2.5}$$

and thus the light-cone Hamiltonian

$$H_{\mathrm{l.c.}} = p_- \dot{q}^- + \int_0^{2\pi} d\sigma\, \Pi_i \dot{X}^i - L\,. \tag{2.6}$$

Use the mode expansion and the results from the lectures to show that

$$p^- = \frac{1}{2\alpha' p_+} H_{\mathrm{l.c.}}\,. \tag{2.7}$$

3 In this question, you will consider closed string theory with the usual boundary conditions for the first 25 coordinates

$$X^{\hat{\mu}}(\tau, \sigma + 2\pi) = X^{\hat{\mu}}(\tau, \sigma)\,, \quad \hat{\mu} = 0, \ldots, 24\,, \tag{3.1}$$

but modified boundary condition for the 26th coordinate. In each case, write down the mode expansion for X^{25}, the mass-shell and level-matching conditions. We will see some of these different cases arising throughout the lectures.

(a) Consider a string winding w times around a circle of radius R in the 25th direction, i.e.

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi w R\,, \tag{3.2}$$

with $w \in \mathbb{Z}$ and R a constant. Interpret the terms depending on w in the mass formula. *Hint:* What condition should you impose on the momentum p^{25}?

(b) Consider a so-called twisted string state in an orbifold compactification.

$$X^{25}(\tau, \sigma + 2\pi) = -X^{25}(\tau, \sigma)\,. \tag{3.3}$$

Note: the orbifold corresponds to identifying the x^{25} spacetime direction as $x^{25} \sim -x^{25}$.

4 Consider *open* strings, so that now $\sigma \in (0, \pi)$, with Neumann boundary conditions at $\sigma = 0, \pi$ for the first 25 directions,

$$\partial_\sigma X^{\hat{\mu}}(\tau, 0) = \partial_\sigma X^{\hat{\mu}}(\tau, \pi) = 0\,, \qquad \hat{\mu} = 0, \ldots, 24\,, \tag{4.1}$$

and the following boundary conditions at $\sigma = 0, \pi$ for the 26th direction. In each case, write down the mode expansion and compute the classical mass formula by imposing

$$\partial_+ X \cdot \partial_+ X = 0\,, \tag{4.2}$$

and interpret the mass formula. *Hint:* Recall that the spacetime momenum is defined as

$$P_\mu = T \int_0^\pi \dot{X}_\mu\,. \tag{4.3}$$

(a) Dirichlet boundary conditions at $\sigma = 0, \pi$, i.e.

$$X^{25}(\tau, 0) = x_1\,, \qquad X^{25}(\tau, \pi) = x_2\,. \tag{4.4}$$

(b) Mixed (Neumann-Dirichlet) boundary conditions at $\sigma = 0, \pi$, i.e.

$$X^{25}(\tau, 0) = x_1\,, \qquad \partial_\sigma X^{25}(\tau, \pi) = 0\,. \tag{4.5}$$

5 Consider the "twisted" ζ-function

$$\zeta(s, \theta) = \sum_{n=1}^{\infty} (n - \theta)^{-s}\,. \tag{5.1}$$

Regularise $\zeta(-1, \theta)$ like in the lectures to show that the finite part of $\zeta(-1, \theta)$ is given by

$$\zeta(-1, \theta) = -\frac{1}{12}\left(6\theta^2 - 6\theta + 1\right)\,. \tag{5.2}$$