Introduction to String Theory

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Exercise Sheet 5

1 Consider the open string momentum

$$P^\mu = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \, \dot{X}^\mu \,. \tag{1.1}$$

(a) Compute this for open strings with N-N boundary conditions.

(b) Compute this for open strings with D-D boundary conditions. What does this result imply for D-branes?

2 Consider the mode expansion for open strings with N-N, D-D, N-D and D-N boundary conditions.

(a) Show that

$$\partial_\pm X^\mu = \begin{cases} \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma^\pm} & \text{(NN)} \\ \mp\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma^\pm} & \text{(DD)} \\ \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma^\pm} & \text{(ND)} \\ \mp\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma^\pm} & \text{(DN)} \end{cases} \,, \tag{2.1}$$

where n is taken to run over appropriate values and $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$.

(b) Show that all four boundary conditions allow us to define $\partial_+ X^\mu$ on a worldsheet of double the length $0 \leq \sigma \leq 2\pi$ with

$$\partial_+ X^\mu = \begin{cases} \partial_+ X^\mu(\tau, \sigma) \,, & 0 \leq \sigma \leq \pi \,, \\ \pm\partial_- X^\mu(\tau, 2\pi - \sigma) \,, & \pi \leq \sigma \leq 2\pi \,, \end{cases} \tag{2.2}$$

with + sign for (NN) and (DN) and − sign for (DD) and (ND) boundary conditions. This is called the "doubling trick".

Therefore, we see $T_{++} = (\partial_+ X)^2$ and $T_{--} = (\partial_- X)^2$ are not independent. It is useful to define the open string Virasoro generators as

$$L_n = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \left(T_{++} \, e^{in\sigma^+} + T_{--} \, e^{in\sigma^-} \right) \,. \tag{2.3}$$

(c) Show that, using the "doubling trick", the Virasoro generators can equivalently be written as

$$L_n = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, T_{++} \, e^{in\sigma^+} \,. \tag{2.4}$$

(d) Show that in terms of the mode expansion, the Virasoro generators are

$$L_n = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m \,. \tag{2.5}$$

3 In this question, you will construct some open string states at higher levels in light-cone quantisation. Consider (NN) boundary conditions throughout this question for simplicity.

(a) Argue that the open string states at level 1 cannot form representations under $\mathrm{SO}(D-1)$, the massive little group, but only of $\mathrm{SO}(D-2)$, the massless little group.

(b) Construct the open string states at level 2 in and determine their representation under $\mathrm{SO}(D-1)$.

(c) Construct the states at level 3 and show that they fit into a traceless symmetric 3-tensor and an anti-symmetric 2-tensor representation of $\mathrm{SO}(D-1)$.

4 So far we have only considered oriented string theories, where there is an orientation on the worldsheet and thus a distinction between left-/right-movers for closed strings and between the endpoints $\sigma = 0, \pi$ for the open string. In this question, you will explore the unoriented strings.

(a) Show that even after gauge fixing, the classical worldsheet action is invariant under the discrete parity transformation

$$\tau \longrightarrow \tau, \qquad \sigma \longrightarrow \ell - \sigma\,, \tag{4.1}$$

where $\ell = 2\pi$ for the closed string and $\ell = \pi$ for the open string. This is called the orientifold symmetry of the string. In the quantum theory, it is implemented via a unitary operator Ω acting on string fields as

$$X^\mu(\tau,\sigma) \longrightarrow \Omega^\dagger X^\mu(\tau,\sigma)\Omega = X^\mu(\tau, \ell-\sigma)\,. \tag{4.2}$$

(b) Show that the orinetifold symmetry induces the following action on the string modes:

$$\begin{array}{ll} \text{closed} & \Omega^\dagger \alpha_n^\mu \Omega = \tilde{\alpha}_n^\mu\,, \qquad \Omega^\dagger \tilde{\alpha}_n^\mu \Omega = \alpha_n^\mu\,, \\ \text{NN} & \Omega^\dagger \alpha_n^\mu \Omega = (-1)^n \alpha_n^\mu\,, \\ \text{DD} & \Omega^\dagger \alpha_n^\mu \Omega = (-1)^{n+1} \alpha_n^\mu\,, \qquad \Omega^\dagger x_{0/1} \Omega = x_{1/0}\,, \\ \text{DN} & \Omega^\dagger \alpha^\mu_{n+\frac{1}{2}} \Omega = i(-1)^n \alpha^\mu_{n+\frac{1}{2}} \text{ of ND}\,. \end{array} \tag{4.3}$$

The unoriented (or orientifolded) string theory contains only those states of the string spectrum which are invariant under the orientifold $\mathbb{Z}_2$ action (4.2). This requires knowledge of the phase of the action of Ω on the vacuum, which can be constrained but not with techniques you have met at this stage in the course.
For the closed string, there is only one consistent phase of Ω acting on the vacuum:

$$\Omega\,|0;p\rangle = |0;p\rangle\ . \tag{4.4}$$

For the open string, there are two consistent phases:

$$\Omega\,|0;p\rangle = \pm\,|0;p\rangle\ . \tag{4.5}$$

(c) Using the phase given above (4.4), which states does the closed unoriented string theory contain at the first excited level? Using the phases ± 1 of (4.5), which states do open unoriented strings ending on a single D-brane contain?

(d) Consider open strings ending on a stack of N coincident D-branes. Depending on the phase (4.5), the orientifold action exchanges Chan-Paton factors as

$$\Omega\,|0;p;m,n\rangle = \pm\,|0;p;n,m\rangle\ , \tag{4.6}$$

where $m, n = 1, \ldots, N$. Which states are kept at the first excited level for the two signs? Can you guess the corresponding gauge groups?