Introduction to String Theory

Humboldt-Universität zu Berlin
Dr. Emanuel Malek

Exercise Sheet 6

1 In the lectures on CFT so far, we have been studying conformal field theories defined on the complex plane $\mathbb{C}$. However, the conformal field theory of the closed string is defined on the worldsheet Σ which is an infinitely-long cylinder, i.e. the coordinates on Σ are $\sigma \sim \sigma + 2\pi$, $\tau \in (-\infty, +\infty)$. In this exercise, you will connect these two pictures. First, perform a Wick rotation of the coordinates such that $\tau \longrightarrow i\,\tau$ is real and define the complex coordinates on the cylinder

$$w = \sigma + i\,\tau\,, \qquad \sigma \in [0, 2\pi)\,. \tag{1.1}$$

(a) Consider the map $z : \Sigma \longrightarrow \mathbb{C}$ from the cylinder to the complex plane defined as

$$z = e^{-iw}\,. \tag{1.2}$$

Is this a conformal map? Draw lines of constant σ and lines of constant τ in $\mathbb{C}$. Where are the infinite past $\tau \to -\infty$ and the infinite future $\tau \to +\infty$ mapped to?

(b) What does the worldsheet Hamiltonian $l_0 + \tilde{l}_0 = \partial_\tau$ get mapped to on $\mathbb{C}$? What does the worldsheet translation operator $l_0 - \tilde{l}_0 = \partial_\sigma$ get mapped to on $\mathbb{C}$?

(c) If operators are time ordered on the worldsheet cylinder, what does this imply for their ordering on the complex plane?

(d) How do primary fields of weight $(h, \tilde{h})$ on the cylinder transform under the map (1.2) from the worldsheet cylinder to $\mathbb{C}$?

(e) Consider the theory of a free scalar field defined on the cylinder, Σ,

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\, \partial_\alpha X \partial^\alpha X\,. \tag{1.3}$$

Write the mode expansion for X on the cylinder. As we saw in the lectures, ∂X is a primary field of weight $(h, \tilde{h}) = (1, 0)$. Calculate how ∂X transforms under the map (1.2) to the complex plane. What is the mode expanion of ∂X on $\mathbb{C}$?

2 **(a)** Show, using Stoke's theorem, that in two dimensions $\partial^2 \ln{(\sigma - \sigma')^2} = 4\pi\,\delta(\sigma - \sigma')$.

(b) Show that Stoke's Theorem in 2 dimensions in complex coordinates is given by

$$\int_R d^2z\left(\partial v^z + \bar{\partial} v^{\bar{z}}\right) = i\oint_{\partial R}\left(v^z d\bar{z} - v^{\bar{z}} dz\right)\,. \tag{2.1}$$

(c) Verify that in complex coordinates $\partial\bar{\partial}\ln|z|^2 = 2\pi\,\delta(z, \bar{z})$.

3 Show that $:e^{ikX}:$ is a primary operator for the theory of a free scalar field and compute its weight.

4 A theory of a free scalar field X has OPE

$$\partial X(z)\,\partial X(w) = -\frac{\alpha'}{2}\frac{1}{(z-w)^2} + \ldots \tag{4.1}$$

Consider the stress-energy tensor

$$T(z) = -\frac{1}{\alpha'} :\partial X(z)\partial X(z): - Q\,\partial^2 X(z)\,, \tag{4.2}$$

for some constant Q.

(a) Compute the $T\,X$ OPE and determine the transformation of X under conformal transformations.

(b) Show that ∂X has weight $h = 1$ but is not primary unless $Q = 0$. Show that $:e^{ikX}:$ is primary and compute its weight.

(c) How can it be that the theory of a free scalar field can have different stress-energy tensors?

5 Consider a theory of several free, non-interacting scalars, X^μ, $\mu = 1, \ldots, D$ with action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X^\nu\,\eta_{\mu\nu}\,, \tag{5.1}$$

with $\eta_{\mu\nu}$ the Minkowski metric.

(a) Compute the propagator

$$\langle X^\mu(\sigma) X^\nu(\sigma')\rangle\,. \tag{5.2}$$

(b) The stress-energy tensor is given by

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu(z)\partial X_\mu(z):\,. \tag{5.3}$$

Consider the operators

$$\zeta_\mu :\partial X^\mu e^{ik\cdot X}: \quad \text{and} \quad \zeta_{\mu\nu} :\partial X^\mu \bar\partial X^\nu e^{ik\cdot X}:, \tag{5.4}$$

where ζ_μ, k_μ are constant vectors and $\zeta_{\mu\nu}$ is a constant tensor. What are the conditions for these operators to be primary and what are their weights?

6 A free Majorana fermion in two dimensions has the action

$$S = \frac{1}{2\pi}\int d^2z\,\psi\bar\partial\psi + \bar\psi\partial\bar\psi\,. \tag{6.1}$$

The propagator is given by the OPE

$$\psi(z)\psi(w) = -\psi(w)\psi(z) = \frac{1}{z-w}\,, \tag{6.2}$$

and similarly for $\bar\psi$. Remember that ψ and $\bar\psi$ are Grassman-valued fields, i.e. they anticommute. The energy momentum tensor is

$$T_{zz} \equiv T = -\frac{1}{2} :\psi\partial\psi:\,. \tag{6.3}$$

(a) Show that ψ is a primary operator of weight $h = \frac{1}{2}$.

(b) Compute the TT OPE.