Introduction to String Theory

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Exercise Sheet 7

1 Recall the mode expansions we saw in Exercise Sheet 4, question 3b) for the closed string and question 4) for the open string. Use light-cone quantisation to compute the mass-shell formula for these strings.

2 Integrate up the infinitesimal transformation law for the stress-energy tensor

$$\delta T(w) = -\epsilon(w)\,\partial T(w) - 2\,\partial\epsilon(w)\,T(w) - \frac{c}{12}\partial^3\epsilon(w)\,, \tag{2.1}$$

as follows.

(a) By writing the effect of the conformal transformation $z \to w$ as

$$T(z) \to T'(w) = \left(\frac{\partial w}{\partial z}\right)^{-2}\left(T(z) - \frac{c}{12}\{w,z\}\right)\,, \tag{2.2}$$

derive the following transformation properties of $\{w,z\}$:

- $\{z+\epsilon, z\} = \epsilon'''(z) + O(\epsilon^2)$, where $\epsilon'(z) = \partial\epsilon(z)$,
- $\{u,z\} = \left(\frac{\partial w}{\partial z}\right)^2 \{u,w\} + \{w,z\}$,
- $\{w,z\} = -\left(\frac{\partial w}{\partial z}\right)^2 \{z,w\}$.

(b) Using the above, show that

$$\delta\{w,z\} \equiv \{w+\delta w, z\} - \{w,z\} = \left(w'\right)^2 \partial_w^3 \delta w\,, \tag{2.3}$$

where $w' = \partial_z w$.

(c) Use the chain rule to show that

$$\{w,z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2\,. \tag{2.4}$$

3 The bc ghost system plays a crucial rule in the path integral quantisation of the string. It consists of two free Grassman fields b and c. Their OPE is given by

$$b(z)\,c(w) = -c(w)\,b(z) = \frac{1}{z-w}\,, \tag{3.1}$$

and the stress-tensor is

$$T =: (\partial b)\,c : -\lambda\,\partial : bc : \,. \tag{3.2}$$

Show that b and c are primary operators with weight $h = \lambda$ and $h = 1-\lambda$, respectively. Show that the central charge is given by

$$c = -12\,\lambda^2 + 12\,\lambda - 2\,. \tag{3.3}$$

Hint: Do not confuse the field c with the central charge.

4 Consider a conserved holomorphic current $j(z)$ with conserved charge

$$Q = \frac{i}{2\pi} \oint j(z) dz \,. \tag{4.1}$$

By considering the commutator with a field $[Q, \phi(w, \bar{w})]$, rederive the Ward identity

$$\delta\phi(w, \bar{w}) = -\text{Res}_{z \to w} j(z)\phi(w, \bar{w}) \,. \tag{4.2}$$

5 Let $|\partial^m X\rangle$ denote the state isomorphic to the operator $\partial^m X$. Show that for $n > 0$, $\alpha_n |\partial^m X\rangle = 0$ unless $m = n$.

6 Consider the CFT for the open string with complex coordinate $w = \sigma + i\,\tau$, $\sigma \in [0, \pi]$ for the open string worldsheet Σ.

(a) What is the image of the following conformal map?

$$z = e^{-iw} \tag{6.1}$$

What is the string boundary mapped to? What are the boundary conditions mapped to?

(b) Consider the state-operator map. Recall that for the closed string, there is an isomorphism between local operators at the origin of the complex plane and states. For the open string there is a similar isomorphism between local operators and states. Do the local operators in this isomorphism live in the bulk or boundary of the open string?

(c) Determine the propagator $\langle X(z, \bar{z}) X(w, \bar{w}) \rangle$ for the open string with the different possible boundary conditions.

(d) Use the "doubling trick" of Exercise Sheet 5 to define the stress-energy tensor for the open string on the whole complex plane. How are the Virasoro generators defined?

(e) Show that the NN open string operator $: e^{ipX}(w, \bar{w}) :$ is primary when it is on the boundary, i.e. $w = \bar{w}$ and compute its weight.