Introduction to String Theory

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Exercise Sheet 8

1 In this exercise, you will develop the general framework of BRST quantisation. Consider a gauge theory with fields ϕ_i, symmetry parameters ϵ^α and a gauge-fixing condition $F^A(\phi) = 0$, where the indices i, α and A are generalised indices and include labels for the coordinates. In this notation,

$$\phi_i \phi'^i \equiv \int dx \phi_{\hat{i}}(x) \phi'^{\hat{i}}(x) \,, \tag{1.1}$$

where $\hat{i}$ is the actual label for the fields, and the sum over the coordinate part of the i index becomes the integral in (1.1). The gauge symmetry defines the algebra

$$[\delta_\alpha, \, \delta_\beta] = f^\gamma{}_{\alpha\beta} \delta_\gamma \,, \tag{1.2}$$

with structure constants $f^\gamma{}_{\alpha\beta}$.

The path integral for a gauge-invariant theory can be written as

$$Z = \frac{1}{\mathrm{Vol}_{\mathrm{gauge}}} \int D\phi_i e^{-S(\phi)} = \int D\phi_i DB_A Db_A Dc^\alpha e^{-S_1(\phi) - S_2(B) - S_3(b,c)} \,, \tag{1.3}$$

where $S_1(\phi) = S(\phi)$ is the original gauge-invariant action and

$$\begin{aligned} S_2(B) &= -iB_A F^A(\phi) \,, \\ S_3(b,c) &= b_A c^\alpha \delta_\alpha F^A(\phi) \,, \end{aligned} \tag{1.4}$$

are the gauge-fixing and ghost actions, respectively. Note that B_A acts simply as a Lagrange multiplier imposing the gauge-fixing condition

$$F^A(\phi) = 0 \,, \tag{1.5}$$

whereas the b, c fields are Grassmann-valued fields, generalising the b, c ghosts that you will meet in the lectures.

(a) Consider the global, fermionic symmetry

$$\begin{aligned} \delta_{\mathrm{BRST}} \phi_i &= -i\theta c^\alpha \delta_\alpha \phi_i \,, \\ \delta_{\mathrm{BRST}} B_A &= 0 \,, \\ \delta_{\mathrm{BRST}} b_A &= \theta B_A \,, \\ \delta_{\mathrm{BRST}} c^\alpha &= \frac{i}{2} \theta f^\alpha{}_{\beta\gamma} c^\beta c^\gamma \,, \end{aligned} \tag{1.6}$$

where ϵ is a Grassmann-valued symmetry parameter. This is known as the *BRST* symmetry. Show that

$$\delta_{\mathrm{BRST}} \delta'_{\mathrm{BRST}} \psi = 0 \,, \tag{1.7}$$

when $\psi \in \{\phi_i, \, B_A, \, b_A, \, c^\alpha\}$.
Hint: Use the Jacobi identity for the gauge symmetry algebra.

(b) Show that

$$\delta_{\mathrm{BRST}}\delta'_{\mathrm{BRST}}G(\phi, B, b, c) = 0\,, \tag{1.8}$$

for a general functional $G(\phi, B, b, c)$ of the fields ϕ_i, B_A, b_A, c^α. Define the fermionic conserved charge Q associated to the BRST symmetry and argue that

$$Q^2 = 0\,. \tag{1.9}$$

(c) Show that

$$\delta_{\mathrm{BRST}}(b_A F^A) = i\theta\left(S_2 + S_3\right)\,, \tag{1.10}$$

and hence show that the total action $S_1 + S_2 + S_3$ is invariant under the BRST symmetry.

(d) Consider an infinitesimal change of the gauge-fixing condition δF^A. Use (1.10) to show that under this change δF^A, an amplitude

$$\langle f|i\rangle = \int_i^f D\phi_i DB_A Db_A Dc^\alpha e^{-S_1(\phi)-S_2(B)-S_3(b,c)}\,, \tag{1.11}$$

changes by

$$\theta\delta\langle f|i\rangle = -\theta\langle f|\left\{Q,\, b_A\delta F^A\right\}|i\rangle\,. \tag{1.12}$$

Using $Q^\dagger = Q$, argue that this implies that for physical states we must have

$$Q\,|\text{physical}\rangle = 0\,. \tag{1.13}$$

(e) Given any any $|\chi\rangle$, show that the state $Q|\chi\rangle$ is physical, but also orthogonal to all physical states. Therefore, all physical amplitudes involving the null state $Q|\chi\rangle$ vanish. Moreover, two physical states which differ by a null state $Q|\chi\rangle$, i.e.

$$|\text{phys}'\rangle = |\text{phys}\rangle + Q|\chi\rangle\,, \tag{1.14}$$

will have the same inner product with all physical states. As a result, the states in (1.14) are physicall equivalent.

In summary, all physical states satisfy $Q|\text{phys}\rangle = 0$ (i.e. lie in the kernel of Q) and are equivalent if they differ by a state in the image of Q. In other words, the physical but inequivalent states of the system are to be identified with the BRST-cohomology.

2 Now, we will apply the BRST formalism to quantise the point particle. The gauge symmetry is the reparameterisation symmetry

$$\tau \longrightarrow \tau'(\tau)\,. \tag{2.1}$$

As a basis for the infinitesimal transformations, we can use

$$\delta_{\tau_1}\tau = \delta(\tau - \tau_1)\,. \tag{2.2}$$

An infinitesimal transformation is then recovered as

$$\delta_\epsilon\tau = \epsilon^\alpha\delta_\alpha\tau = \int d\tau_1\epsilon(\tau_1)\delta(\tau-\tau_1) = \epsilon(\tau)\,. \tag{2.3}$$

The reparameterisation symmetry acts on the fields $\phi_i = (X^\mu(\tau),\, e(\tau))$ as

$$\begin{aligned} \delta_{\tau_1} X^\mu(\tau) &= -\delta(\tau-\tau_1)\partial_\tau X^\mu(\tau)\,, \\ \delta_{\tau_1} e(\tau) &= -\partial_\tau\left[\delta(\tau-\tau_1)e(\tau)\right]\,. \end{aligned} \tag{2.4}$$

(a) Show that (2.4) recovers the usual transformation law for X^μ and e under infinitesimal reparameterisations.
(b) Show that the algebra of reparameterisations has structure constants

$$f^{\tau_3}{}_{\tau_1\tau_2} = \delta(\tau_3-\tau_1)\partial_{\tau_3}\delta(\tau_3-\tau_2) - \delta(\tau_3-\tau_2)\partial_{\tau_3}\delta(\tau_3-\tau_1)\,. \tag{2.5}$$

(c) Show that the BRST symmetry (1.6) reduces to

$$\begin{aligned} \delta_{\rm BRST} X^\mu(\tau) &= i\theta c(\tau)\partial_\tau X^\mu(\tau)\,, \\ \delta_{\rm BRST} e(\tau) &= i\theta\partial_\tau\left(c(\tau)e(\tau)\right)\,, \\ \delta_{\rm BRST} B(\tau) &= 0\,, \\ \delta_{\rm BRST} b(\tau) &= \theta B(\tau)\,, \\ \delta_{\rm BRST} c(\tau) &= i\theta c(\tau)\partial_\tau c(\tau)\,. \end{aligned} \tag{2.6}$$

(d) Show that the full action $S_1+S_2+S_3$ with the gauge fixing condition $F(\tau) = 1-e(\tau)$ is given by

$$S_1+S_2+S_3 = \int d\tau\left(\frac{1}{2}e^{-1}\dot{X}^2 + \frac{1}{2}em^2 + iB\,(e-1) - e\dot{b}c\right)\,, \tag{2.7}$$

with $\dot{X} = \partial_\tau X$, etc. Integrate out B and impose the constraint from the e equation of motion to find the modified BRST symmetry

$$\begin{aligned} \delta_{\rm BRST} X^\mu(\tau) &= i\theta c\dot{X}^\mu\,, \\ \delta_{\rm BRST} b(\tau) &= i\theta\left(\frac{1}{2}(m^2-\dot{X}^2) - \dot{b}c\right)\,, \\ \delta_{\rm BRST} c(\tau) &= i\theta c\dot{c}\,. \end{aligned} \tag{2.8}$$

Under what conditions does this symmetry square to zero?

(e) Show that $Q = c\,H$, with H the Hamiltonian

$$H = \frac{1}{2}\left(p^2+m^2\right)\,, \tag{2.9}$$

and where $p^\mu = i\dot{X}^\mu$ is the Euclidean canonical momentum.

(f) The b, c ghosts have canonical anticommutators

$$\{b,\, c\} = 1\,, \tag{2.10}$$

and thus generate a two-state system $|\uparrow\rangle$, $|\downarrow\rangle$. A complete set of states is $|k,\uparrow\rangle$, $|k,\downarrow\rangle$ with

$$\begin{aligned} p^\mu\,|k,\downarrow\rangle &= k^\mu\,|k,\downarrow\rangle\,, & p^\mu\,|k,\uparrow\rangle &= k^\mu\,|k,\uparrow\rangle\,, \\ b\,|k,\downarrow\rangle &= 0\,, & b\,|k,\uparrow\rangle &= |k,\downarrow\rangle\,, \\ c\,|k,\downarrow\rangle &= |k,\uparrow\rangle\,, & c\,|k,\uparrow\rangle &= 0\,. \end{aligned} \tag{2.11}$$

What is the spectrum of physically inequivalent states?