Introduction to String Theory

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Exercise Sheet 9

1 Consider the b, c ghost action

$$S_{gh} = \frac{1}{2\pi} \int d^2\sigma \sqrt{g} \left(b_{\alpha\beta} \nabla^\alpha c^\beta \right) . \tag{1.1}$$

Recall that $b_{\alpha\beta}$ is traceless, i.e. $b_{\alpha\beta} g^{\alpha\beta} = 0$. Compute the stress-energy tensor as follows.

(a) Show that under a metric variation, the Christoffel symbols transform as

$$\delta\Gamma^\alpha{}_{\beta\gamma} = -g_{\rho(\beta} \nabla_{\gamma)} \delta g^{\alpha\rho} + \frac{1}{2} g_{\beta\rho} g_{\gamma\sigma} \nabla^\alpha \delta g^{\rho\sigma} . \tag{1.2}$$

(b) Introduce a Lagrange multiplier into the action (1.1) to enforce tracelessness of $b_{\alpha\beta}$. Now vary the action with respect to $g^{\alpha\beta}$ to show that

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S_{gh}}{\delta g^{\alpha\beta}} = g_{\alpha\beta} b_{\gamma\delta} \nabla^\gamma c^\delta - 2 b_{\gamma(\alpha} \nabla_{\beta)} c^\gamma + c^\gamma \nabla_\gamma b_{\alpha\beta} . \tag{1.3}$$

Confirm that (1.3) is traceless as required by Weyl invariance.
Hint: To find $T_{\alpha\beta}$, you need to solve for the Lagrange multiplier.

(c) Show that (1.3) reduces to the stress-energy tensor met in the lectures in flat gauge:

$$\begin{aligned} T &= 2\,(\partial c)\, b + c\partial b\,, \\ \bar{T} &= 2\left(\bar{\partial}\bar{c}\right) \bar{b} + \bar{c}\bar{\partial}\bar{b}\,. \end{aligned} \tag{1.4}$$

2 Apply the BRST formalism to the string, as you did in question 2 of exercise sheet 8 for the point particle. In particuar, consider infinitesimal diffeomorphisms and Weyl transformations in a basis such that

$$\begin{aligned} \delta_{\sigma_1^\alpha} X^\mu &= -\delta\left(\sigma_1^\alpha - \sigma^\alpha\right) \partial_\alpha X^\mu\,, \\ \delta_\omega X^\mu &= 0\,, \\ \delta_{\sigma_1^\alpha} g_{\beta\gamma} &= -\delta\left(\sigma_1^\alpha - \sigma^\alpha\right) \partial_\alpha g_{\beta\gamma} - g_{\alpha\gamma} \partial_\beta \delta(\sigma_1^\alpha - \sigma^\alpha) \\ &\quad - g_{\alpha\beta} \partial_\gamma \delta(\sigma_1^\alpha - \sigma^\alpha)\,, \\ \delta_\omega g_{\alpha\beta} &= -2 g_{\alpha\beta}\,, \end{aligned} \tag{2.1}$$

where there is no sum over repeated indices.

(a) Confirm that this gives the right transformation laws for infinitesimal diffeomorphisms

$$\begin{aligned} \delta_\epsilon X^\mu &= \int d^2\sigma_1 \epsilon^\alpha(\sigma_1) \delta_{\sigma_1^\alpha} X^\mu\,, \\ \delta_\epsilon g_{\alpha\beta} &= \int d^2\sigma_1 \epsilon^\gamma(\sigma_1) \delta_{\sigma_1^\gamma} g_{\alpha\beta}\,. \end{aligned} \tag{2.2}$$

(b) Show that the only non-zero structure constants for the gauge transformations are given by

$$f^{\sigma_3^\gamma}{}_{\sigma_1^\alpha \sigma_2^\beta} = \delta(\sigma_3^\alpha - \sigma_1^\alpha)\partial_\alpha \delta(\sigma_3^\beta - \sigma_2^\beta)\delta_\beta^\gamma - \delta(\sigma_3^\beta - \sigma_2^\beta)\partial_\beta \delta(\sigma_3^\alpha - \sigma_1^\alpha)\delta_\alpha^\gamma \,, \tag{2.3}$$

where there is no sum over α or β.

(c) Show that the BRST transformations reduce to

$$\begin{aligned} \delta_{\mathrm{BRST}} X^\mu &= i\theta c^\alpha \partial_\alpha X^\mu \,, \\ \delta_{\mathrm{BRST}} g_{\alpha\beta} &= i\theta \left(c^\gamma \partial_\gamma g_{\alpha\beta} + 2 g_{\gamma(\alpha} \partial_{\beta)} c^\gamma + 2\tilde{c} g_{\alpha\beta} \right) \,, \\ \delta_{\mathrm{BRST}} b_{\alpha\beta} &= \theta B^{\alpha\beta} \,, \\ \delta_{\mathrm{BRST}} B^{\alpha\beta} &= 0 \,, \\ \delta_{\mathrm{BRST}} c^\alpha &= i\theta c^\beta \partial_\beta c^\alpha \,, \\ \delta_{\mathrm{BRST}} \tilde{c} &= 0 \,, \end{aligned} \tag{2.4}$$

where c^α are the reparameterisation ghosts and $\tilde{c}$ is the Weyl ghost.

(d) Use the gauge fixing condition $F_{\alpha\beta} \to g_{\alpha\beta} - \delta_{\alpha\beta}$ to show that the ghost and gauge fixing actions are given by

$$\begin{aligned} S_3 &= -2 \int d^2\sigma \sqrt{g} \left(b^{\alpha\beta} \nabla_\alpha c_\beta + b^{\alpha\beta} g_{\alpha\beta} \tilde{c} \right) \,, \\ S_2 &= i \int d^2\sigma \sqrt{g} B^{\alpha\beta} (g_{\alpha\beta} - \delta_{\alpha\beta}) \,. \end{aligned} \tag{2.5}$$

(e) Integrate out the $B^{\alpha\beta}$ field and use the $g_{\alpha\beta}$ equation of motion to simplify the BRST transformations to

$$\begin{aligned} \delta_{\mathrm{BRST}} X^\mu &= i\theta c^\alpha \partial_\alpha X^\mu \,, \\ \delta_{\mathrm{BRST}} c^\alpha &= i\theta c^\beta \partial_\beta c^\alpha \,, \\ \delta_{\mathrm{BRST}} b_{\alpha\beta} &= -i\theta \left(T^{\alpha\beta}_{\mathrm{poly}} + T^{\alpha\beta}_{\mathrm{ghost}} \right) \,. \end{aligned} \tag{2.6}$$

3 Consider the state corresponding to the operator

$$\mathcal{O}(z,\bar{z}) = : c(z)\bar{c}(\bar{z}) V^{(m)}(z,\bar{z}) : \,, \tag{3.1}$$

where $V^{(m)}$ is an operator of the matter CFT.

(a) Show that if $V^{(m)}$ is a primary operator of weight $(h,\tilde{h}) = (1,1)$, then the state corresponding to $\mathcal{O}(z,\bar{z})$ is BRST-closed.

(b) Show that if $V^{(m)}$ is a primary operator of weight $(h,\tilde{h}) = (1,1)$, then the operator

$$\mathcal{V} = \int d^2z V^{(m)}(z,\bar{z}) \,, \tag{3.2}$$

is also BRST-closed. Interpret this in terms of conformal invariance of the operator $\mathcal{V}(z,\bar{z})$.

(c) What are the conditions for

$$\mathcal{O}(z,\bar{z}) = :c(z)\bar{c}(\bar{z})\zeta_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ip\cdot X}(z,\bar{z}):, \tag{3.3}$$

where $\zeta_{\mu\nu}$ is a constant "polarisation" tensor, to be BRST-closed?

(d) Consider the operator

$$\lambda(z,\bar{z}) = :\bar{c}(\bar{z})\zeta_\nu \bar{\partial} X^\nu e^{ip\cdot X}(z,\bar{z}):, \tag{3.4}$$

with ζ a constant tensor satisfying $p\cdot\zeta = 0$. Show that

$$[Q_B, \lambda(z,\bar{z})] = i : c(z)\bar{c}(z) p_\mu \zeta_\nu \partial X^\mu \bar{\partial} X^\nu e^{ip\cdot X}(z,\bar{z}) : . \tag{3.5}$$

What does this result imply for the polarisation tensor $\zeta_{\mu\nu}$ in (3.3).

(e) Consider the operators (3.3) corresponding to the graviton, Kalb-Ramond field and the dilaton, respectively. What does the above results imply for the gauge symmetries of these fields?